Numerical Methods for Differential Equations FMNN10/NUMN12
Exam 2013-12-16 Results to be announced 2013-12-23

Exam duration 14:00 – 19:00. A minimum of 16 points out of 32 are required to pass. Your grade is determined by the sum of your exam and project scores, in accordance with the rules on the course home page.

No computers, pocket calculators, cell phones, browsing tablets or any other electronic devices, and no textbooks, lecture notes or written material, may be used during the exam.

1. **(5p)** If the initial value problem $\dot{y} = f(y)$ is solved by the explicit midpoint method, with initial conditions $y_0 = y(0)$ and y_1, we have the recursion

$$y_{n+1} - y_{n-1} = 2hf(y_n).$$

   (a) Determine the order of consistency of the method. (2p)

   (b) Apply the method to the linear test equation $\dot{y} = \lambda y$ and determine the method's stability region. (3p)

2. **(5p)** Consider an implicit Runge-Kutta method with Butcher tableau

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1/3 & 2/3 \\ \hline b^{\mathrm{T}} & 1/3 & 2/3 \end{array}$$

   (a) Apply the method to the linear test equation $\dot{y} = \lambda y$ with $y(0) = 1$ and find the method's stability function $R(h\lambda)$. (3p)

   (b) Is the method A-stable? (2p)

3. **(4p)** Consider the following nonlinear two-point boundary value problem:

$$\begin{aligned} &y'' - (y^2)' + y = g(x) \\ &y(0) = \alpha, \qquad y(1) = \beta. \end{aligned}$$

   (a) Introduce a suitable grid and discretize with a standard second order method. Give all details about the grid (number of grid points and their location, as well as mesh width Δx) and formulate the discretization. Include the boundary conditions in the equation system. (2p)

(b) Construct the Jacobian matrix associated with the system. (2p)

4. **(5p)** Two students, **A** and **B**, want to solve the eigenvalue problem

$$u'' + u = \lambda u$$

with one Dirichlet boundary condition $u(0) = 0$ and one Robin condition $u(1) + u'(1) = 0$ on $[0, 1]$.

Student A decides to introduce a grid with $x_j = j\Delta x$, taking $\Delta x = 1/(N + 1/2)$, and then approximates the Robin condition by

$$\frac{u_{N+1} + u_N}{2} + \frac{u_{N+1} - u_N}{\Delta x} = 0.$$

Solving for u_{N+1}, student **A** gets

$$u_{N+1} = \frac{2 - \Delta x}{2 + \Delta x} u_N$$

and proceeds to construct a second order method, based on an $N \times N$ tridiagonal matrix A. Because **A** uses the discretization

$$\frac{u_{j-1} - 2u_j + u_{j+1}}{\Delta x^2} + u_j = \lambda_{\Delta x} u_j,$$

the *last row* of the matrix A, which accounts for the Robin condition, reads

$$\begin{pmatrix} 0 \ldots 0 & \dfrac{1}{\Delta x^2} & \dfrac{-2 - 3\Delta x + 2\Delta x^2 + \Delta x^3}{\Delta x^2 (2 + \Delta x)} \end{pmatrix}$$

Student B, however, insists that one can take a grid with $x_j = j\Delta x$, taking $\Delta x = 1/(N + 1)$ so that $x_{N+1} = 1$, and still obtain a second order method. **B** approximates the derivative at $x = 1$ by the second order BDF2 method:

$$u'(1) \approx \frac{1}{\Delta x} \left(\frac{3}{2} u_{N+1} - 2u_N + \frac{1}{2} u_{N-1} \right).$$

Student **B** then proceeds with exactly the same discretization as **A**, except taking $\Delta x = 1/(N + 1)$ and representing the Robin boundary condition using the BDF2 approximation. **B** constructs a linear algebraic eigenvalue problem $Bu = \lambda u$.

(a) Show that student **B**'s BDF2 approximation is second order accurate, i.e., that

$$u'(1) = \frac{1}{\Delta x} \left(\frac{3}{2} u_{N+1} - 2u_N + \frac{1}{2} u_{N-1} \right) + \mathrm{O}(\Delta x^2).$$

(2p)

(b) In the approximation of the Robin condition $u(1) + u'(1) = 0$, solve for u_{N+1} in terms of u_N and u_{N-1}. (1p)

(c) Construct the last row of student **B**'s matrix B. (2p)

5. **(5p)** For $t \geq 0$ and $x \in [0,1]$, let u_j^n approximate $u(j \cdot \Delta x, n \cdot \Delta t)$. Write down the *differential equations* corresponding to the explicit finite difference discretizations below, and give *the name of the equation* in each case.

(a)
$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + u_j^n \cdot \frac{u_{j+1}^n - u_j^n}{\Delta x} = 0$$

(b)
$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{u_{j-1}^n - 2u_j^n + u_{j+1}^n}{\Delta x^2} + f(u_j^n)$$

(c)
$$\frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{\Delta t^2} = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}$$

(d)
$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} - \Delta t \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{2\Delta x^2} = 0$$

(e)
$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{u_{j+1}^n - u_j^n}{\Delta x} + \frac{u_{j-1}^n - 2u_j^n + u_{j+1}^n}{\Delta x^2}$$

6. **(4p)**

(a) Show that any differentiable function $u(x)$ on $[0,1]$, satisfying the "periodic boundary condition" $u(0) = u(1)$, has the property $\langle u, u' \rangle = 0$, where

$$\langle u, v \rangle = \int_0^1 u(x)v(x)\,\mathrm{d}x.$$

(2p)

(b) Conclude that in the advection equation $u_t = u_x$ with periodic boundary conditions $u(t,0) = u(t,1)$, the norm $\|u(t,\cdot)\|_2$ of the solution remains constant for all t. Here $\|u(t,\cdot)\|_2^2 = \langle u, u \rangle$. (2p)

7. **(4p)** Consider the convection–diffusion equation

$$u_t = u_{xx} + u_x$$

with homogeneous boundary conditions, and initial condition $u(0, x) = g(x)$.

(a) Introduce a suitable notation and write down a standard 2nd order method-of-lines discretization in space combined with the *trapezoidal rule* for time-stepping ("Crank–Nicolson's method"). Give the details about your grid and choice of Δx. (2p)

(b) As the method is implicit, one will have to solve a linear system of equations on each step. Construct this system. (2p)

LYCKA TILL!
G.S.