LUNDS TEKNISKA HÖGSKOLA
MATEMATIK

LÖSNINGAR
LINJÄR OCH KOMBINATORISK OPTIMERING
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1. a) We start by multiplying the second constraint by -1 so that the right hand side of all constraints is nonnegative. Then we introduce slack variables x_4 and x_5 to the second and the third constraints to put the problem on canonical form. The slack variables x_4 and x_5 can be used as basic variables for a first basic feasible solution, but we need one more basic variable, and to find one, we introduce an artificial variable y_1 in the second equation. (It would also have been correct to use three artificial variables, one for each constraint.) The objective is then to minimize y_1 subject to these constraints or, equivalently, to maximize $z = -y_1$. The LP problem that should be solved in phase 1 is therefore

$$\begin{array}{ll} \text{maximize} & z = -y_1 \\ \text{subject to} & \left\{ \begin{array}{rcrcrcrcrcrcl} 3x_1 & + & x_2 & - & 2x_3 & + & x_4 & & & & & = & 2, \\ -x_1 & + & 4x_2 & & & & & & & + & y_1 & = & 8, \\ 4x_1 & - & x_2 & + & x_3 & & & + & x_5 & & & = & 5, \\ x_1, & & x_2, & & x_3, & & x_4, & & x_5, & & y_1 & \geq & 0, \end{array} \right. \end{array}$$

In the tableau, we need z to be expressed in terms of the nonbasic variables x_1, x_2 and x_3, and so the equation in the objective row should not be $z + y_1 = 0$, but instead $z + x_1 - 4x_2 = -8$. The first tableau is therefore

	x_1	x_2	x_3	x_4	x_5	y_1	
x_4	3	1	-2	1	0	0	2
y_1	-1	4	0	0	0	1	8
x_5	4	-1	1	0	1	0	5
z	1	-4	0	0	0	0	-8

b) We have to choose x_2 as the incoming variable, and then either x_4 or y_1 can be outgoing variables (since the quotients $2/1$ and $8/4$ are the same. We choose for example x_4 as an outgoing variable. The next tableau is then

	x_1	x_2	x_3	x_4	x_5	y_1	
x_2	3	1	-2	1	0	0	2
y_1	-13	0	8	-4	0	1	0
x_5	7	0	-1	1	1	0	7
z	13	0	-8	4	0	0	0

c) Yes, since there is a solution with the artificial variable $y_1 = 0$, there is a feasible solution for the original problem, and it is given by the nonbasic variables $x_1 = 0$, $x_4 = 0$ and the basic variables $x_2 = 2$, $x_3 = 0$, $x_5 = 7$. The first three rows of the tableau will be the same as the final tableau from phase 1 except that the y_1 row will be removed. The objective row comes from $z = x_1 - 3x_2 + x_3$, when expressed in only the nonbasic variables x_1 and x_4, i e $z = \frac{15}{8}x_1 + \frac{1}{2}x_4 - 6$. The first tableau of phase 2 is therefore

	x_1	x_2	x_3	x_4	x_5	
x_2	$-1/4$	1	0	0	0	2
x_3	$-13/8$	0	1	$-1/2$	0	0
x_5	$43/8$	0	0	$1/2$	1	7
z	$-15/8$	0	0	$-1/2$	0	-6

d) Either x_1 or x_4 can be chosen as incoming variables, and we choose e.g. x_4 (following the advice to make the calculations as simple as possible). The outgoing variable has to be x_5. The next tableau is

	x_1	x_2	x_3	x_4	x_5	
x_2	$-1/4$	1	0	0	0	2
x_3	$15/4$	0	1	0	1	7
x_4	$43/4$	0	0	1	2	14
z	$7/2$	0	0	0	1	1

Since all the coefficients in the objective row are nonnegative, the optimality condition is satisfied. We have found an optimal solution $\mathbf{x} = (0, 2, 7, 14, 0)^T$ with the optimal value $z = 1$.

2. The statement will be proved if we can show that the LP problem

$$\begin{array}{ll} \text{maximize} & z = 7x_1 + 5x_2 \\ \text{subject to} & \begin{cases} 2x_1 - x_2 \leq 3 \\ 3x_1 + x_2 \geq 9 \\ -x_1 + 4x_2 \leq 16 \\ x_1, \quad x_2 \geq 0 \end{cases} \end{array}$$

has an optimal solution with optimal value $z \leq 53$.
Drawing the domain in the x_1x_2-plane, we see that the domain will be as in Figure 1 below, where we have also indicated a level curve (line) at the optimum.

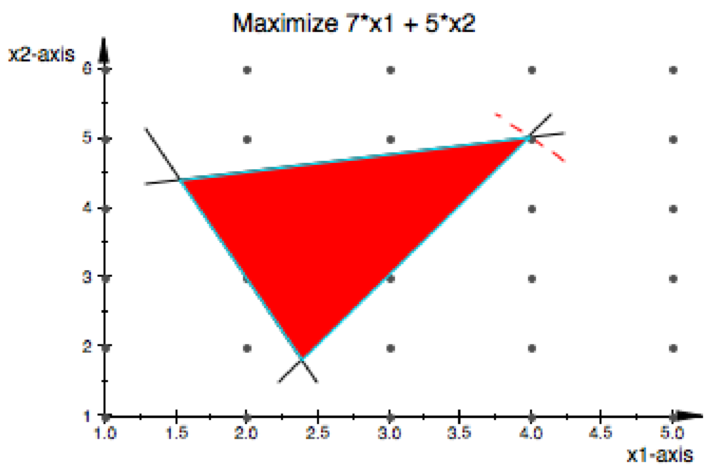

Figure 1: The domain and optimum of the LP problem

Clearly, the maximum value is $z = 7 \cdot 4 + 5 \cdot 5 = 53$, and we are done.

3. a) We need to check that the sum of the incoming capacities are the same as the sum of the outgoing capacities for all the nodes except for s and t, and that the source node s has only outgoing capacities and the sink node has only incoming capacities. The flow on each edge also has to be nonnegative and less than the capacity for that edge. This clearly holds for the flow f. The value of the flow is the sum of the outgoing capacitÄies of the source node, or, equivalently, the sum of the incoming capacities of the sink node, and so the flow f has the value 5.

b) See the digraph below for the excess capacities. In the digraph, the number written to the left of the arrow from node i to node j is the excess capacity d_{ij}, and the number written to the right of the same arrow is the excess capacity d_{ji} in the opposite direction. Using the Ford–Fulkersson method, we also find a path with capacity 3 as indicated with the fatter edges in the same network below:

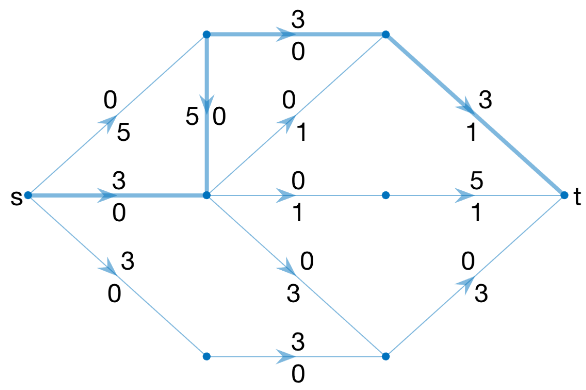

Figure 2: The excess capacities of the network given the flow f and a path from s to t

The next figure shows the digraph with the new flow with value 8:

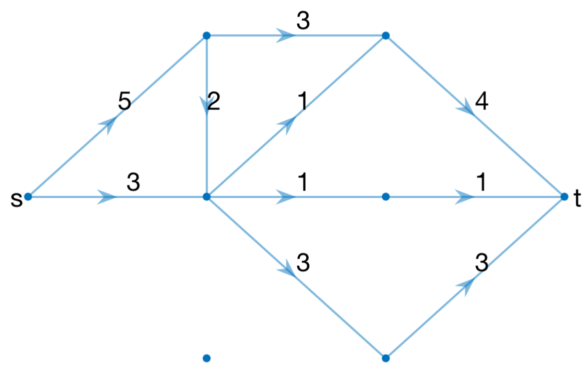

Figure 3: A new flow for the network with value 8

When we update the network with new excess capacities, we obtain the following figure:

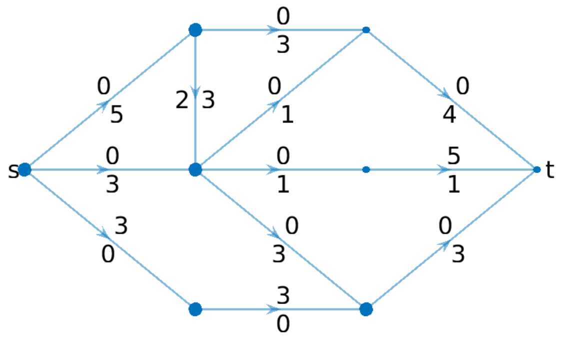

Figure 4: The network with the updated excess capacities

The nodes that can be labelled in this stage of the algorithm have been highlighted. As you can see, two nodes, including t, cannot be labelled. The Ford–Fulkersson method terminates here.

c) According to the max flow-min cut theorem, the set of directed edges from the unlabelled to the labelled nodes form a minimum cut. See the figure with the original network together with a minimum cut:

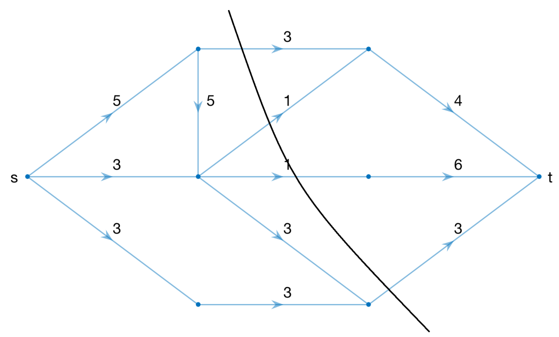

Figure 5: The original network with a minimum cut of capacity 8

As you can see, the capacity of the cut is 8, as expected by the max flow-min cut theorem.

4. a) Each of the inequality constraints have the form $a_1x_1 + \ldots a_nx_n \leq b$ and involves n multiplications, $n-1$ additions and one comparison, giving $2n$ operations for each constraint, and so, checking that the solution is feasible requires $2nm$ operations. For the number of steps of the simplex algorithm, I will accept an estimate of number of elementary computations (of the right order).

The optimization criterion in the simplex algorithm is $\mathbf{c_B^T A_B^{-1} A_N} - \mathbf{c_N^T} \geq \mathbf{0}$, using the notation from the course. To verify this condition, we need to compute $A_B^{-1}A_N$, which is typically done by Gauss elimination. Solving this as a system $A_BX = A_N$, where A_B has size $m \times m$, and A_N and the unknown matrix X have size $m \times n$. Hence the augmented matrix $[A_B|A_N]$ has size $m \times (n+m)$.

The number of operations required for the Gauss elimination is of order $\mathcal{O}((n+m)m^2)$: To make the argument as simple as possible, I will just give an upper bound of that order. An upper bound for the number of operations needed for making a pivot element equal to 1 is $n+m$, and since there are m pivot elements in the matrix, an upper bound for creating all these 1s is $(n+m)m$. For each zero created when we put the matrix in upper triangular form, we need at most $2(n+m)$ operations, and the number of zeros in this matrix is $m \times (m+1)/2$. We also need to create the same number of zeros when doing the back substitution. We can then calculate an upper bound for the Gauss elimination to $(n+m)m + 2(n+m)m(m+1) = (n+m)m(1+2m+2)$.
Note that this was the number of operations needed for computing $A_B^{-1}A_N$. Then we need to multiply this $m \times n$-matrix from the left by $\mathbf{c_B}$ and subtracting $\mathbf{c_N^T}$ to get the vector $\mathbf{c_B^T A_B^{-1} A_N} - \mathbf{c_N^T}$, and for this, $m \times n$ multiplications and n subtractions are needed. Finally, checking that this vector has only nonnegative entries requires n comparisons. Hence we get totally an upper bound of $nm(1+2m+2)+(n+m)(m+2) = \mathcal{O}((n+m)m^2)$.

b) The definition of NP is that a solution can be checked with a polynomial time algorithm, and from subproblem a), we have a polynomial time algorithm for checking that a given solution is feasible and optimal. Yes, LP belongs to P since there exist polynomial time algorithms for finding an optimal solution, e.g. the ellipsoid method or central path methods (but not the simplex algorithm). (0.5)

5. We introduce a slack variable x_5 with the constraint $x_5 \in \{0,1,2,3,4,5,6,7\}$ to make the constraint an equality constraint. The directed graph with the labels are as in the figure below:

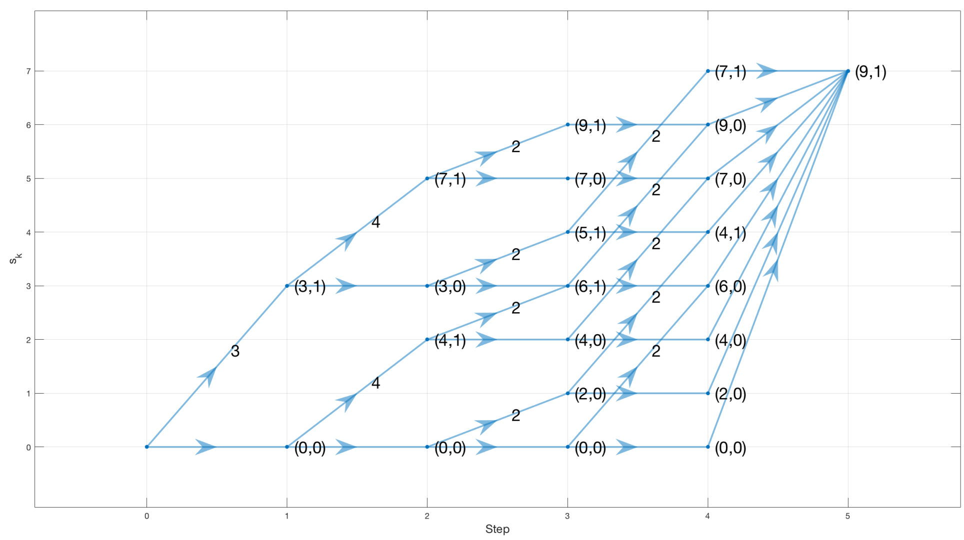

Figure 6: The digraph for the knapsack problem

The maximum value is 9, and it is achieved if $x_1 = 1$, $x_2 = 1$, $x_3 = 1$, $x_4 = 0$ (and $x_5 = 1$).

6. We note that there is no demand associated with customer C_4. Instead customer C_4 is offering to buy any left-over of the product, but C_3 is offering the same thing. The sensible thing for the production company to do is to sell what is left over to the one of C_3 and C_4 that gives the highest benefit. Hence, anything left over from P_1 or P_3 should be sold to C_4, while the left over from P_2 should be sold to C_3. We note that the total supply is 500 units, while the total demand is 450 units. Hence, the left over that will be sold to C_3 or C_4 is 50 units of the product. We get the actual cost matrix together with the supply and demand vectors for which the total supply is equal to the total demand.

	C_1	C_2	C_3	C_3/C_4	Supply
P_1	60	40	45	55	130
P_2	70	55	65	65	200
P_3	80	60	55	75	170
Demand	150	175	125	50	

This problem is equivalent to the one in the problem formulation, and it can be solved with the transportation algorithm (with maximization instead of minimization). We fill in the table using a maximum benefit rule (the rule corresponding to the minimum cost rule for maximization problems).

	C_1	C_2	C_3	C_3/C_4	
P_1	60	40	45	55	
	130	0	0	0	130
P_2	70	55	65	65	
	20	55	125	0	200
P_3	80	60	55	75	
	0	120	0	50	170
	150	175	125	50	

The reduced cost matrix has zeros in the places corresponding to the nonzero values in the matrix above, and then the dual variables can be computed according using $\widehat{c}_{ij} = c_{ij} - v_i - w_j$ for these i and j giving $v_1 = 0$, $v_2 = 10$, $v_3 = 15$, $w_1 = 60$, $w_2 = 45$, $w_3 = 55$, $w_4 = 60$. The rest of the reduced costs can now be computed according to $\widehat{c}_{ij} = c_{ij} - v_i - w_j$, and they are as in the table below:

0	−5	−10	−5
0	0	0	−5
5	0	−15	0

The interpretation of the reduced cost $\widehat{c}_{ij}$ is the change in the objective function value if the variable x_{ij} would increase with one unit. Since we want the objective value to increase, we need to increase the x_{31}. When doing this, we need to decrease x_{32} and x_{21} and increase x_{22} by the same amount. The maximal change that is possible is 20, because then x_{21} becomes 0. We get an updated transportation scheme as follows:

	C_1	C_2	C_3	C_3/C_4	
P_1	60 130	40 0	45 0	55 0	130
P_2	70 0	55 75	65 125	65 0	200
P_3	80 20	60 100	55 0	75 50	170
	150	175	125	50	

The updated values for the dual variables are $v_1 = 0$, $v_2 = 30$, $v_3 = 20$, $w_1 = 60$, $w_2 = 40$, $w_3 = 35$, $w_4 = 55$, and the new reduced costs are given in the table

0	0	10	0
−20	0	0	−20
0	0	0	0

To increase the benefit, we need to change x_{13}, and then x_{23}, x_{22}, x_{32}, x_{31} and x_{11} will also change. The maximal amount we can change it with is 100, when x_{32} becomes 0. The new transportation scheme is

	C_1	C_2	C_3	C_3/C_4	
P_1	60 30	40 0	45 100	55 0	130
P_2	70 0	55 175	65 25	65 0	200
P_3	80 120	60 0	55 0	75 50	170
	150	175	125	50	

The updated dual variables are $v_1 = 0$, $v_2 = 20$, $v_3 = 20$, $w_1 = 60$, $w_2 = 35$, $w_3 = 45$ and $w_4 = 55$. The reduced costs are

0	5	0	0
−10	0	0	−10
0	5	−10	0

In order to increase the benefit, we can change x_{12} or x_{32}. I will change x_{12}, and then also x_{13}, x_{23} and x_{22} will change. The maximum possible increase of x_{12} is 100, and after making this change, the transportation scheme will look as follows:

	C_1	C_2	C_3	C_3/C_4	
P_1	60	40	45	55	
	30	100	0	0	130
P_2	70	55	65	65	
	0	75	125	0	200
P_3	80	60	55	75	
	120	0	0	50	170
	150	175	125	50	

We compute new values for the dual values, and they are $v_1 = 0$, $v_2 = 15$, $v_3 = 20$, $w_1 = 60$, $w_2 = 40$, $w_3 = 50$ and $w_4 = 55$. The table with reduced costs is as follows:

0	0	-5	0
-5	0	0	-15
0	0	-15	0

Since all reduced costs are nonpositive, we have found an optimal solution with value 31400.

In order to maximize the benefit, the producer should sell 30 units from P_1 to customer C_1 and 40 units from P_2 to customer C_2. From P_2 they should sell 75 units to C_2 and 125 units to C_3. Finally, they should sell 120 units from P_3 to C_1 and 50 units to C_4.