LUNDS TEKNISKA HÖGSKOLA
MATEMATIK

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LINJÄR OCH KOMBINATORISK OPTIMERING
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1. a) The first thing to do is to rewrite the problem so that the right hand side of all constraints are positive. This yields the system

$$\begin{array}{ll} \text{maximize} & z = x_1 + 3x_2 - 2x_3 \\ \text{subject to} & \left\{ \begin{array}{rcrcrcl} 2x_1 & + & 3x_2 & - & x_3 & \leq & 2, \\ -x_1 & + & 4x_2 & & & = & 5, \\ -7x_1 & + & 5x_2 & + & 2x_3 & \geq & 4, \\ x_1, & & x_2, & & x_3 & \geq & 0. \end{array} \right. \end{array}$$

To write the problem in canonical form, we need two slack variables, x_4 and x_5. The system is then

$$\begin{array}{ll} \text{maximize} & z = x_1 + 3x_2 - 2x_3 \\ \text{subject to} & \left\{ \begin{array}{rcrcrcrcrcl} 2x_1 & + & 3x_2 & - & x_3 & + & x_4 & & & = & 2, \\ -x_1 & + & 4x_2 & & & & & & & = & 5, \\ -7x_1 & + & 5x_2 & + & 2x_3 & & & - & x_5 & = & 4, \\ x_1, & & x_2, & & x_3, & & x_4, & & x_5 & \geq & 0. \end{array} \right. \end{array}$$

For phase 1, we need (at least) two artificial variables, y_1 and y_2, and the objective function for phase 1 is $y_1 + y_2$, which has to be minimized, or equivalently, $-y_1 - y_2$, which has to be maximized. The LP problem in canonical form for phase 1 is

$$\begin{array}{ll} \text{maximize} & z = -y_1 - y_2 \\ \text{subject to} & \left\{ \begin{array}{rcrcrcrcrcrcrcl} 2x_1 & + & 3x_2 & - & x_3 & + & x_4 & & & & & & & = & 2, \\ -x_1 & + & 4x_2 & & & & & & & + & y_1 & & & = & 5, \\ -7x_1 & + & 5x_2 & + & 2x_3 & & & - & x_5 & & & + & y_2 & = & 4, \\ x_1, & & x_2, & & x_3, & & x_4, & & x_5, & & y_1, & & y_2 & \geq & 0. \end{array} \right. \end{array}$$

To write down the first tableau, we need to express the objective function in terms of the nonbasic variables x_1, x_2, x_3 and x_5 (the basic variables are the artificial variables y_1 and y_2 together with the slack variable x_4):

$$z = -y_1 - y_2 = -(5 + x_1 - 4x_2) - (4 + 7x_1 - 5x_2 - 2x_3 + x_5) = -8x_1 + 9x_2 + 2x_3 - x_5 - 9,$$

and so the first tableau of phase 1 is

	x_1	x_2	x_3	x_4	x_5	y_1	y_2	$\mathbf{x_b}$
x_4	2	3	-1	1	0	0	0	2
y_1	-1	4	0	0	0	1	0	5
y_2	-7	5	2	0	-1	0	1	4
z	8	-9	-2	0	1	0	0	-9

b) Choosing the variable with the most negative coefficient (for example), x_2 will be incoming and x_4 will be outgoing. The new tableau is

	x_1	x_2	x_3	x_4	x_5	y_1	y_2	$\mathbf{x_b}$
x_2	$2/3$	1	$-1/3$	$1/3$	0	0	0	$2/3$
y_1	$-11/3$	0	$4/3$	$-4/3$	0	1	0	$7/3$
y_2	$-31/3$	0	$11/3$	$-5/3$	-1	0	1	$2/3$
z	14	0	-5	3	1	0	0	-3

c) Yes, since the optimal objective value for phase 1 is 0, there exists a (basic) feasible solution of the original problem. We find one by choosing x_2, x_5 and x_3 as basic variables and removing the columns corresponding to the y coordinates. The objective function has to be expressed in the nonbasic variables:

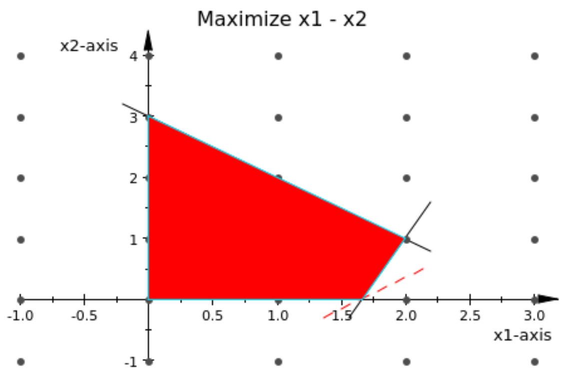
$$z = x_1 + 3x_2 - 2x_3 = x_1 + 3\left(\frac{5}{4} + \frac{1}{4}x_1\right) - 2\left(\frac{7}{4} + \frac{11}{4}x_1 + x_4\right) = -\frac{15}{4}x_1 - 2x_4 + \frac{1}{4},$$

and so we can write the first tableau of phase 2 as

	x_1	x_2	x_3	x_4	x_5	$\mathbf{x_b}$
x_2	$-1/4$	1	0	0	0	$5/4$
x_5	$1/4$	0	0	-2	1	$23/4$
x_3	$-11/4$	0	1	-1	0	$7/4$
z	$15/4$	0	0	2	0	$1/4$

d) The tableau shows that the solution that we found in subproblem c) is optimal (since all the coefficients in the objective row are nonnegative). The optimal solution is $\mathbf{x} = (0, 5/4, 7/4, 0, 23/4)^T$, and the optimal value is $1/4$.

2. a) The feasible set is the set that is marked in red in the figure below. The level curves of the objective function are lines that are parallel to the dashed line in the figure, and the value gets higher by going down/right.

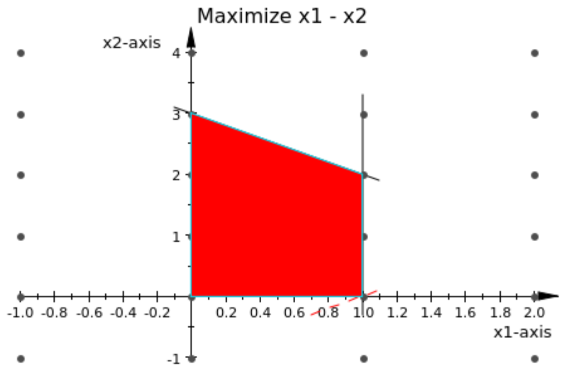


The optimal value is at the intersection point of the lines defined by $x_2 = 0$ and $3x_1 - x_2 = 5$, i.e. $(x_1, x_2) = (5/3, 0)$. The optimal value of the relaxed problem is $5/3$.

b) The branching variable is x_1, and we divide the feasible set into two parts by intersecting it with the half planes $x_1 \leq 1$ and $x_1 \geq 2$, respectively. For the left branch (corresponding to $x_1 \leq 1$), we solve the problem

$$\begin{array}{ll} \text{maximize} & z = x_1 - x_2 \\ \text{subject to} & \left\{ \begin{array}{rcrcl} 3x_1 & - & x_2 & \leq & 5, \\ x_1 & + & x_2 & \leq & 3, \\ x_1 & & & \leq & 1, \\ & & x_1, \; x_2 & \geq & 0. \end{array} \right. \end{array}$$

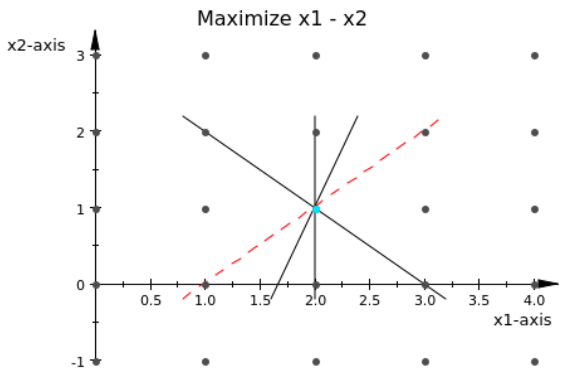
By drawing the domain and the level set of the objective function corresponding to the optimal value, we see that the optimal point is $(x_1, x_2) = (1, 0)$ and the optimal value is 1:



We note that this is an integer solution.
Next, we turn to the right branch (corresponding to $x_2 \geq 2$). The LP problem to solve is

$$\begin{array}{ll} \text{maximize} & z = x_1 - x_2 \\ \text{subject to} & \left\{ \begin{array}{rcrcl} 3x_1 & - & x_2 & \leq & 5, \\ x_1 & + & x_2 & \leq & 3, \\ x_1 & & & \geq & 2, \\ & & x_1, \; x_2 & \geq & 0, \end{array} \right. \end{array}$$

whose feasible set is the single point $(2, 1)$:



The optimal solution is therefore $(x_1, x_2) = (2, 1)$, and the optimal value is 1. We note that this is also an integer solution.
Hence both branches give the same value 1, which is attained in the two points $(1, 0)$ and $(2, 1)$.

3. a) Let c_{ij} $i, j = 1, \dots, n$ be the elements of a given cost matrix. The assignment problem is to find $x_{ij} \in \{0, 1\}$ which maximizes (or minimizes) $z = \sum_{ij} c_{ij} x_{ij}$ subject to the constraints

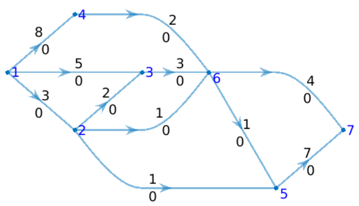
$$\begin{cases} \displaystyle\sum_{j=1}^{n} x_{ij} = 1, & i = 1, \dots, n, \\ \displaystyle\sum_{i=1}^{n} x_{ij} = 1, & j = 1, \dots, n. \end{cases}$$

b) Let c'_{ij} be the entries of the matrix $\mathbf{C}'$. We assume that $\mathbf{C}'$ is formed by adding α to row number k of $\mathbf{C}$. Then

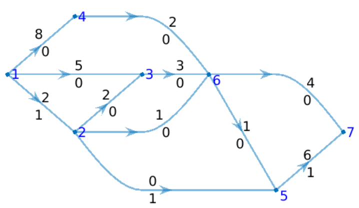
$$\begin{aligned} \sum_{ij} c'_{ij} x_{ij} &= \sum_{\substack{ij \\ i \neq k}} c_{ij} x_{ij} + \sum_{i} (c_{ik} + \alpha) x_{ik} \\ &= \sum_{ij} c_{ij} x_{ij} + \alpha \sum_{i} x_{ik} = \sum_{ij} c_{ij} x_{ij} + \alpha, \end{aligned}$$

where we used the constraint $\sum_i x_{ik} = 1$ in the last equality. The maximum (or minimum) of $\sum_{ij} c'_{ij} x_{ij}$ must therefore be attained precisely when (x_{ij}) is a maximum (or minimum) of $\sum_{ij} c_{ij} x_{ij}$.
In exactly the same way one can prove the statement when α is added to a column instead of a row. The proof is complete.

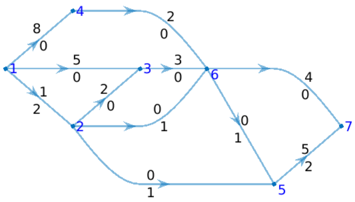
4. a) Start with the initial flow which is 0 on all edges. We compute the excess capacities. They are indicated in the network below:



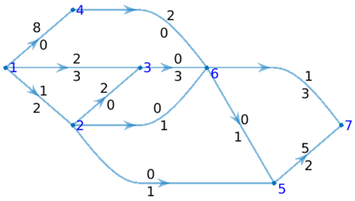
Following the Ford–Fulkerson algorithm, we start labelling nodes that can be reached from the start node. In the first step we label node number 2, 3 and 4 with the labels $(3, 1)$, $(5, 1)$ and $(8, 1)$, respectively, where the first number is the excess capacity of the path and the second number is the predecessor index. Node number 5 and 6 can be reached from node 2, and they will both be labelled with $(1, 2)$. Node 7 can now be labelled with $(1, 5)$. We have found an augmenting path: $1 \to 2 \to 5 \to 7$ with excess capacity 1. The flow is increased by 1 along that path, and after computing the new excess capacities, we find that the network is as follows:



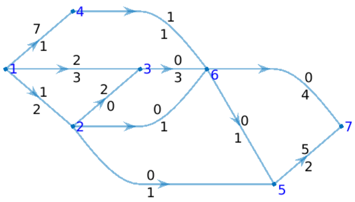
Next, we label the nodes with numbers 2, 3, 4, 6, 5, 7 with the labels $(2,1)$, $(5,1)$, $(8,1)$, $(1,2)$, $(1,6)$ and $(1,5)$, respectively. This gives the path $1 \to 2 \to 6 \to 5 \to 7$ with excess capacity 1. We compute a new flow and new excess capacities and obtain the following network:



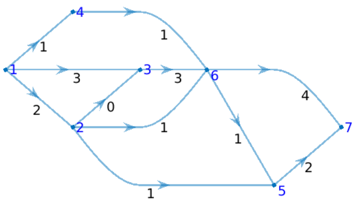
Next, we can label the nodes with numbers 2, 3, 4, 6 and 7 with the labels $(1,1)$, $(5,1)$, $(8,1)$, $(3,3)$ and $(3,6)$, respectively, giving the augmenting path $1 \to 3 \to 6 \to 7$ with excess capacity 3. When the new flow and excess capacities have been computed, we get the network



Now we label the nodes number 2, 3, 4, 6 and 7 with the labels $(1,1)$, $(2,1)$, $(8,1)$, $(2,4)$ and $(1,6)$, respectively, and we get the augmenting path $1 \to 4 \to 6 \to 7$ with excess capacity 1. The updated network is



Labelling the nodes ones more, we find that we cannot find a path from node 1 to node 7. The flow we have found has value 6 and is given by the network



where we have labelled the edges with x_{ij}. To show that this flow is indeed optimal, note that there is a cut with value 6, and it is defined by the two sets $\{1,2,3,4,6\}$ and $\{5,7\}$. By the max flow–min cut theorem, it follows that the flow that was found is optimal.

b) False: Consider for example the network $1 \xrightarrow{2} 2 \xrightarrow{1} 3 \xrightarrow{1} 4$. The value of the maximum flow is clearly 1, and it is not possible to increase the value of the flow by just changing the capacity of one of the edges.

5. After introducing slack variables x_{41}, x_{42} and x_{43}, we see that the problem corresponds to the transportation problem with cost matrix

	W_1	W_2	W_3	W_4
F_1	3	10	6	0
F_2	12	5	7	0
F_3	8	6	10	0

with supply and demand vectors $(70, 110, 80)^T$ and $(90, 30, 100, 40)^T$, respectively (but note that x_{ij} represents how many units should be shipped from factory number j to warehouse i, and not the other way around as would be the usual way). Now the problem can be solved with the transportation algorithm.

We find an initial transportation scheme by using the minimal cost rule:

	W_1	W_2	W_3	W_4	
F_1	3	10	6	0	
	30	0	0	40	70
F_2	12	5	7	0	
	0	30	80	0	110
F_3	8	6	10	0	
	60	0	20	0	80
	90	30	100	40	

Next, the reduced costs are computed. By complementary slackness $\widehat{c}_{ij} = 0$ if $x_{ij} = 0$. Hence we can partially fill in the reduced cost matrix

$$\widehat{\mathbf{C}} = \begin{pmatrix} 0 & ? & ? & 0 \\ ? & 0 & 0 & ? \\ 0 & ? & 0 & ? \end{pmatrix}.$$

To fill in the rest of the matrix, we use $\widehat{c}_{ij} = c_{ij} - v_i - w_j$ for the c_{ij} which are 0, and set v_1 to 0. We obtain $v_1 = 0$, $v_2 = 2$, $v_3 = 5$, $w_1 = 3$, $w_2 = 3$, $w_3 = 5$ and $w_4 = 0$. The rest of the $\widehat{c}_{ij}$ can be computed and are as follows:

$$\widehat{\mathbf{C}} = \begin{pmatrix} 0 & 7 & 1 & 0 \\ 7 & 0 & 0 & -2 \\ 0 & -2 & 0 & -5 \end{pmatrix}.$$

The smallest negative entry of $\widehat{\mathbf{C}}$ is -5, which indicates that we should increase x_{43}. The maximum increase is 40, and the new shipping scheme is as follows:

	W_1	W_2	W_3	W_4	
F_1	3	10	6	0	
	70	0	0	0	70
F_2	12	5	7	0	
	0	30	80	0	110
F_3	8	6	10	0	
	20	0	20	40	80
	90	30	100	40	

The new reduced cost matrix can now be computed and is given by

$$\widehat{\mathbf{C}} = \begin{pmatrix} 0 & 7 & 1 & 5 \\ 7 & 0 & 0 & 3 \\ 0 & -2 & 0 & 0 \end{pmatrix}.$$

Now x_{32} can be increased by 20. We get the new transportation scheme

	W_1	W_2	W_3	W_4	
F_1	3	10	6	0	
	70	0	0	0	70
F_2	12	5	7	0	
	0	10	100	0	110
F_3	8	6	10	0	
	20	20	0	40	80
	90	30	100	40	

The reduced cost matrix is now

$$\widehat{\mathbf{C}} = \begin{pmatrix} 0 & 9 & 3 & 5 \\ 5 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \end{pmatrix}.$$

Since all its entries are nonnegative, the transportation scheme is optimal. The solution to the original LP problem is $x_{11} = 70$, $x_{12} = 0$, $x_{13} = 20$, $x_{21} = 0$, $x_{22} = 10$, $x_{23} = 20$, $x_{31} = 0$, $x_{32} = 100$ and $x_{33} = 0$, and the value is 1240.

6. The problem can be solved with a slight modification of Dijkstra's algorithm (shortest route problem). The key point is that if p is the probability that information gets lost between two particular stations, then $1 - p$ is the probability that the information is transmitted. The probability that a message is safely transmitted between two stations via a third station is the product of two probabilities (since the events are independent). We think of the stations as nodes in a network, and label the nodes one by one: The first label is the probability that a message is safely transmitted from station A to the current station, using the best route via intermediate stations that have already been labelled. The second label is a predecessor index, i.e. it specifies via what station the message came.

The probability to send a message from A to B directly is 0.95. Node B is labelled with $(0.95, A)$.

The probability to send a message from A to C directly is 0.85 and the probability to send it via node B is $0.95 \cdot 0.90 = 0.855$. Hence C is labelled with $\max(0.85, 0.855) = 0.855$ and B.

Similarly, the first label for node D is $\max(0.70, 0.95 \cdot 0.82, 0.855 \cdot 0.89) = \max(0.70, 0.739, 0.76095) = 0.76095$, and the second label is C.

The first label for node E is $\max(0.65, 0.95 \cdot 0.71, 0.855 \cdot 0.81, 0.76095 \cdot 0.91) = \max(0.65, 0.6745, 0.69255, 0.6916645) = 0.69255$ and the second label is C.

The first label for node F is $\max(0.55, 0.95 \cdot 0.62, 0.855 \cdot 0.72, 0.76095 \cdot 0.80, 0.69255 \cdot 0.98) = \max(0.55, 0.5880, 0.6156, 0.60876, 0.679099) = 0.679099$ and the second label is E.

Using the predecessor index, we see that the optimal way from A to F is $A \to B \to C \to E \to F$. The probability that the message is lost for this route is $1 - 0.679099 = 0.320901$.