

INGA HJÄLPMEDEL/NO TOOLS ARE ALLOWED.

Every problem accounts for a maximum of 1 point. The numbers in brackets near the right margin indicate the maximum points awarded for that subproblem. For passing, a minimum number of 3 points is required, and the grade you get is the integer part of the awarded points (with the exception that 6.0 points gives the grade 5). Careful explanations of your solutions should be provided. Write your solutions in Swedish or English.

1. We would like to solve the LP problem

$$\begin{array}{ll} \text{maximize} & z = x_1 - 3x_2 + x_3 \\ \text{subject to} & \begin{cases} 3x_1 + x_2 - 2x_3 \leq 2, \\ x_1 - 4x_2 = -8, \\ 4x_1 - x_2 + x_3 \leq 5, \\ x_1, x_2, x_3 \geq 0, \end{cases} \end{array}$$

with the two-phase method.

- a) Write down a LP problem on canonical form that should be solved in phase 1. What is the first simplex tableau for phase 1? (0.4)
- b) Make one iteration of the simplex method starting from the tableau in subproblem a). Explain how you choose incoming and outgoing variables. (0.2)
- c) After some more iterations, you find that the final tableau of phase 1 is

| | x_1 | x_2 | x_3 | x_4 | x_5 | y_1 | |
|-------|---------|-------|-------|--------|-------|-------|---|
| x_2 | $-1/4$ | 1 | 0 | 0 | 0 | $1/4$ | 2 |
| x_3 | $-13/8$ | 0 | 1 | $-1/2$ | 0 | $1/8$ | 0 |
| x_5 | $43/8$ | 0 | 0 | $1/2$ | 1 | $1/8$ | 7 |
| z | 0 | 0 | 0 | 0 | 0 | 1 | 0 |

where y_1 is an artificial variable. (It is possible that you used a different number of artificial variables in part a), and this is ok.) Does the original problem have a feasible solution? If so, what should the first tableau of phase 2 be? (0.2)

- d) Solve the LP problem with the simplex method, starting from the tableau you found in subproblem c). Is there an optimal solution? If so, write it down. If the problem is unbounded, explain how you can see this from the tableau.
Hint: Choose the incoming variable so that the computations become as simple as possible. (0.2)

VAR GOD VÄND! / PLEASE TURN OVER!

2. Prove that

$$\begin{cases} 2x_1 - x_2 \leq 3 \\ 3x_1 + x_2 \geq 9 \\ -x_1 + 4x_2 \leq 16 \\ x_1, x_2 \geq 0 \end{cases} \implies 7x_1 + 5x_2 \leq 53$$

by formulating and solving an appropriate linear programming problem.

Hint: Use a geometric method instead of the simplex method.

3. Consider the network N shown in Figure 1 below and the initial flow f from node s to node t as shown in Figure 2.

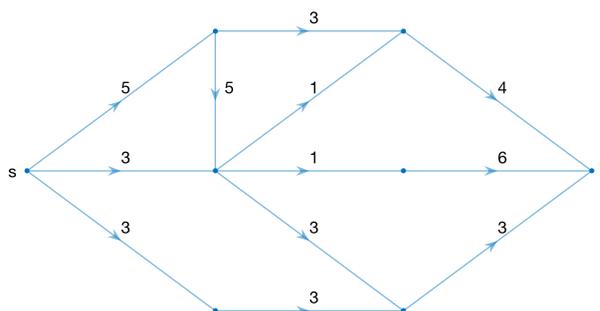


Figure 1: The network N

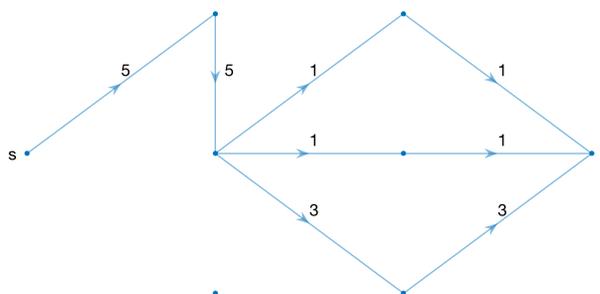


Figure 2: An initial flow f in the network N

- What must be checked to show that f is a flow? What is the value of the flow f ? (0.2)
- Calculate the excess capacities for the flow f in the network N and indicate them in a digraph. Then follow the Ford–Fulkerson method to find updated flows until you have an optimal solution. Draw the digraph of the updated flows (similar to that in Figure 2). What is the value of the maximal flow? (0.6)
- Produce a minimal cut for the network N . (0.2)

VAR GOD VÄND! / PLEASE TURN OVER!

4. a) Consider an LP problem with n variables and m inequality constraints. What is the maximal number of elementary operations ($+$, $-$, \times , \div , \leq) needed for checking that a given solution \mathbf{x} is feasible and optimal? (0.5)
- b) Prove using a) that LP belongs to the class NP . Does LP belong to the class P as well? Motivate your answer. (0.5)
5. Consider the 0/1 knapsack problem

$$\begin{aligned} & \text{maximize } 3x_1 + 4x_2 + 2x_3 + 2x_4 \\ & \text{subject to } \begin{cases} 3x_1 + 2x_2 + x_3 + 3x_4 \leq 7 \\ x_1, x_2, x_3, x_4 \in \{0, 1\}. \end{cases} \end{aligned}$$

Draw the corresponding (directed) graph as a preparation for solving this problem with dynamic programming and a longest (most expensive) route problem. Then use dynamic programming to solve the problem with this method. Make sure that you annotate each node with the node price and steering (0 or 1).

6. An enterprise manufactures a product in three production plants, P_1 , P_2 and P_3 , with a production capacity of 130, 200 and 170 units of product, respectively. The demand of four customers has to be satisfied as follows: customer C_1 demands 150 product units, customer C_2 demands 175, and customer C_3 demands at least 125. Both customers C_3 and C_4 are prepared to buy any spare product units, and they both want to buy as many units of product as possible. The benefit obtained from the sale of units of product to the customers is the following:

| | C_1 | C_2 | C_3 | C_4 |
|-------|-------|-------|-------|-------|
| P_1 | 60 | 40 | 45 | 55 |
| P_2 | 70 | 55 | 65 | 60 |
| P_3 | 80 | 60 | 55 | 75 |

Formulate the matrix format of the transportation problem so as to maximize the total benefit. Then solve the problem with (a maximization version of) the transportation algorithm.

LYCKA TILL! / GOOD LUCK!