LUNDS TEKNISKA HÖGSKOLA
MATEMATIK

TENTAMENSSKRIVNING
LINJÄR OCH KOMBINATORISK OPTIMERING
Exempeltenta

INGA HJÄLPMEDEL/NO TOOLS ARE ALLOWED.
Lösningarna skall vara försedda med ordentliga motiveringar. Skriv dina lösningar på svenska eller engelska.
Careful explanations of your solutions should be provided. Write your solutions in Swedish or English.

1. a) Set up the initial simplex tableau for the LP problem

$$\begin{array}{ll} \text{maximize} & z = x_1 + 3x_2 + 5x_3 \\ \text{subject to} & \begin{cases} 2x_1 & -5x_2 & +x_3 & \leq 3, \\ x_1 & +4x_2 & & \leq 5, \\ x_1, & x_2, & x_3 & \geq 0 \end{cases} \end{array}$$

(0.2)

b) Perform one step in the simplex algorithm using any (legal) rule for choosing incoming and outgoing variable. What does the second tableau look like? (0.4)

c) Solve the rest of the problem with the simplex algorithm. Describe what you do. How do you know when you have found an optimal solution? What is the optimal value (if any)? What optimal solution did you find (if any)? Give your answer using the original variables. (0.4)

2. Consider the following ILP problem:

$$\begin{array}{ll} \text{maximize} & z = x_1 + x_2 \\ \text{subject to} & \begin{cases} 2x_1 & +3x_2 & \leq 12, \\ 2x_1 & +\ x_2 & \leq 6, \\ x_1, & x_2 & \geq 0 \text{ integers.} \end{cases} \end{array}$$

a) When solving the LP relaxed problem (i.e. the problem as above but without integer constraints), you reach the final tableau:

	x_1	x_2	x_3	x_4	$\mathbf{x_b}$
x_1	1	0	$-1/4$	$3/4$	$3/2$
x_2	0	1	$1/2$	$-1/2$	3
z	0	0	$1/4$	$1/4$	$9/2$

Construct an equation for a cutting plane, using the information from this tableau. Construct a tableau for the system with this new constraint added. (0.5)

b) Construct the dual tableau of the tableau you wrote down for the primal problem, to find a feasible solution of the dual system. You do not have to solve the system. (0.5)

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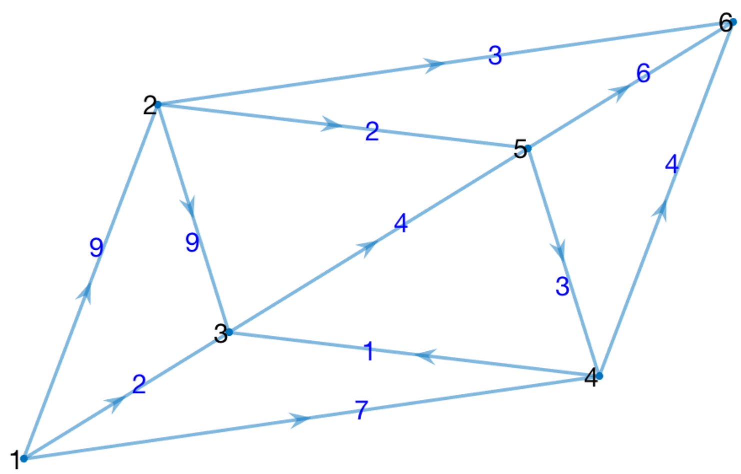
3. Using the definition of the dual of a problem in standard form, write down the dual of the linear programming problem

$$\begin{array}{ll} \text{maximize} & (\mathbf{c}^{\mathbf{T}}\mathbf{x} + \mathbf{d}^{\mathbf{T}}\mathbf{x}') \\ \text{subject to} & \begin{cases} \mathbf{A}\mathbf{x} + \mathbf{A}'\mathbf{x}' \leq \mathbf{b}, \\ \mathbf{x} \geq \mathbf{0}, \\ \mathbf{x}' \in \mathbb{R}. \end{cases} \end{array}$$

Prove your claim and simplify your expressions as much as possible.

4. Consider the flow network below with source $S = 1$ and sink $T = 6$, with edge capacities indicated near each edge:



a) Find a maximum flow in the network. Explain how you came up with this solution. (0.5)

b) Find a minimum cut in the network, i.e. write down two sets of vertices, each belonging to the two nonempty disjoint sets of vertices that define the minimum cut. (0.5)

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5. A company has been contracted for four jobs. These jobs can be performed in five of its manufacturing plants. Because of the size of the jobs, it is not feasible to assign more than one job to a particular manufacturing facility. Also, the second job $J2$ cannot be assigned to the third manufacturing plant $P3$. The cost estimated, in thousands of dollars, of performing the jobs in the different manufacturing plants, are summarized in the table:

	$P1$	$P2$	$P3$	$P4$	$P5$
$J1$	50	55	42	57	48
$J2$	66	70	–	68	75
$J3$	71	78	72	80	85
$J4$	40	42	38	45	46

Naturally, the manufacturer would like the total cost to be as small as possible.

a) Describe what you need to do in order to put the problem in the setting of a standard (minimization) assignment problem. (0.3)

b) Solve the problem with the Hungarian algorithm. (0.7)

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6. A linear fractional optimization is the problem of maximizing or minimizing a quotient of affine functions, i.e. a function of the form

$$\frac{\mathbf{c}^T\mathbf{x}+\alpha}{\mathbf{d}^T\mathbf{x}+\beta},$$

subject to some linear equality or inequality constraints (i.e. the constraint set is a convex polyhedron). Here, $\mathbf{c}$, $\mathbf{d}$ are constant vectors of length n, while α and β are given real numbers (constants). $\mathbf{x}\in\mathbb{R}^n$ is the unknown variable. Clearly, we also have to assume that $\{\mathbf{x};\ \mathbf{d}^T\mathbf{x}+\beta\neq 0\}$ in order for the objective function to be defined.
These types of problems can be solved by considering the cases $\mathbf{d}^T\mathbf{x}+\beta>0$ and $\mathbf{d}^T\mathbf{x}+\beta<0$ separately, and then transforming each of these subproblems to an LP problem using a Charnes–Cooper transformation:

$$\begin{cases} \mathbf{y}=\dfrac{1}{\mathbf{d}^T x+\beta}\mathbf{x}, \\ t=\pm\dfrac{1}{\mathbf{d}^T\mathbf{x}+\beta}, \end{cases}$$

where the sign choice in the equation for t is such that $t>0$.
Consider the problem of minimizing

$$\frac{4x_1+2x_2+5}{2x_1+x_2+2}$$

subject to

$$\begin{cases} 4x_1+3x_2\leq 8, \\ x_1,x_2\geq 0. \end{cases}$$

Note that the denominator of the objective function is always positive for feasible $\mathbf{x}$, and so you don't have to consider subproblems.

a) Transform the problem using a Charnes–Cooper transformation. What is the equivalent LP problem? What is the inverse transformation, i.e. how do you find $\mathbf{x}$ from $\mathbf{y}$ and t? (0.5)

b) Eliminate the variable t from the LP problem (using the equality constraint in the LP problem), and solve the problem graphically, using the gradient of the new objective function in the y_1, y_2 plane. What is the optimal solution? Don't forget to transform the optimal solution back to the original variables. (0.5)

LYCKA TILL! / GOOD LUCK!