LUND UNIVERSITY

Linear Algebra
Thursday, 29 October 2015
Duration: 08:00–13:00

Centre for Mathematical Sciences
Mathematics, Faculty of Science

Solutions

1. We seek to minimise the distance between the vectors $A\boldsymbol{u}$ and $\boldsymbol{v}$ where

$$A = \begin{bmatrix} 4 & -2 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix}, \quad \boldsymbol{u} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \quad \text{and} \quad \boldsymbol{v} = \begin{bmatrix} 4 \\ 2 \\ 1 \\ 3 \\ 5 \end{bmatrix}.$$

This amounts to solving the normal equations

$$A^t A\boldsymbol{u} = \begin{bmatrix} 34 & 0 & 10 \\ 0 & 10 & 0 \\ 10 & 0 & 5 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = A^t\boldsymbol{v} = \begin{bmatrix} 41 \\ 3 \\ 15 \end{bmatrix} \quad \Leftrightarrow \quad \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 11/14 \\ 3/10 \\ 10/7 \end{bmatrix}.$$

Thus the polynomial is $\frac{11}{14}x^2 + \frac{3}{10}x + \frac{10}{7}$.

2. Let $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ be the standard basis for $\mathbf{R}^3$. The third equality immediately gives $F(\boldsymbol{e}_3) = (1,1,1)$. Using the linearity of F and the first two equalities, we get

$$4F(\boldsymbol{e}_2) = F(4\boldsymbol{e}_2) = F(-(1,-2,3) + (1,2,3)) = -F(1,-2,3) + F(1,2,3) = (8,4,8),$$

and hence $F(\boldsymbol{e}_2) = (2,1,2)$. Finally,

$$\begin{aligned} F(\boldsymbol{e}_1) &= F(\boldsymbol{e}_1 - 2\boldsymbol{e}_2 + 3\boldsymbol{e}_3) + 2F(\boldsymbol{e}_2) - 3F(\boldsymbol{e}_3) \\ &= (1,2,3) + 2(2,1,2) - 3(1,1,1) = (2,1,4). \end{aligned}$$

The columns of the matrix are the coordinate vectors of the images of the basis vectors. The matrix is, therefore,

$$\begin{bmatrix} 2 & 2 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix}.$$

3. We can start out with any two linearly independent vectors in the plane, for example $\boldsymbol{v}_1 = (1,-1,0)$ and $\boldsymbol{v}_2 = (1,2,-1)$, and then apply the Gram–Schmidt process to them. We set $\boldsymbol{f}_1 = \boldsymbol{v}_1$ and $\boldsymbol{f}_2 = s\boldsymbol{f}_1 + \boldsymbol{v}_2$ and solve the equation

$$0 = \boldsymbol{f}_1 \cdot \boldsymbol{f}_2 = s\|\boldsymbol{f}_1\|^2 + \boldsymbol{f}_1 \cdot \boldsymbol{v}_2 = 2s - 1 \quad \Leftrightarrow \quad s = \frac{1}{2}.$$

Hence, $\boldsymbol{f}_2 = \frac{1}{2}\boldsymbol{f}_1 + \boldsymbol{v}_2 = \frac{1}{2}(3,3,-2)$ is orthogonal to $\boldsymbol{f}_1$ and lies in the plane. Normalising $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$, we get $\boldsymbol{e}_1 = \frac{1}{\sqrt{2}}(1,-1,0)$ and $\boldsymbol{e}_2 = \frac{1}{\sqrt{22}}(3,3,-2)$. This basis can be

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extended with a unit normal vector $\boldsymbol{e}_3 = \frac{1}{\sqrt{11}}(1,1,3)$ of the plane to a basis for $\mathbf{R}^3$. In summation, $\boldsymbol{e}_1 = \frac{1}{\sqrt{2}}(1,-1,0)$ and $\boldsymbol{e}_2 = \frac{1}{\sqrt{22}}(3,3,-2)$ form an orthonormal basis for the plane. $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3 = \frac{1}{\sqrt{11}}(1,1,3)$ form an orthonormal basis for $\mathbf{R}^3$.

4. The symmetric matrix of the quadratic form is

$$B = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 3 & -1 \\ -1 & -1 & 1 \end{bmatrix}.$$

We have that

$$\det(B - \lambda I) = \begin{vmatrix} 1-\lambda & 1 & -1 \\ 1 & 3-\lambda & -1 \\ -1 & -1 & 1-\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & 1 & -1 \\ 1 & 3-\lambda & -1 \\ -\lambda & 0 & -\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & 1 & \lambda-2 \\ 1 & 3-\lambda & -2 \\ -\lambda & 0 & 0 \end{vmatrix}$$

$$= -\lambda \begin{vmatrix} 1 & \lambda - 2 \\ 3-\lambda & -2 \end{vmatrix} = -\lambda(\lambda-1)(\lambda-4).$$

Hence, the eigenvalues of B are $\lambda_1 = 0$, $\lambda_2 = 1$ and $\lambda_3 = 4$. The minimum value of q is 0 and the maximum value is 4. We can write

$$q(x_1, x_2, x_3) = 0y_1^2 + 1y_2^2 + 4y_3^2$$

where the y_i's are the coordinates of (x_1, x_2, x_3) with respect to an orthonormal basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ of eigenvectors belonging to λ_1, λ_2 and λ_3, respectively. Then $x_1^2+x_2^2+x_3^2 = 1$ if and only $y_1^2+y_2^2+y_3^2 = 1$. The minimum value is assumed at $(y_1, y_2, y_3) = \pm(1,0,0)$ and the maximum value at $(y_1, y_2, y_3) = \pm(0,0,1)$. Hence, we must find $\boldsymbol{e}_1$ and $\boldsymbol{e}_3$. The eigenvectors belonging to λ_1 are given by

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 1 & 3 & -1 & 0 \\ -1 & -1 & 1 & 0 \end{array}\right] \quad\Leftrightarrow\quad \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 2 & 0 & 0 \end{array}\right] \quad\Leftrightarrow\quad \boldsymbol{x} = t(1,0,1)$$

and the eigenvectors belonging to λ_3 by

$$\left[\begin{array}{ccc|c} -3 & 1 & -1 & 0 \\ 1 & -1 & -1 & 0 \\ -1 & -1 & -3 & 0 \end{array}\right] \quad\Leftrightarrow\quad \left[\begin{array}{ccc|c} -2 & 0 & -2 & 0 \\ 1 & -1 & -1 & 0 \\ -2 & 0 & -2 & 0 \end{array}\right] \quad\Leftrightarrow\quad \boldsymbol{x} = t(1,2,-1).$$

We can choose $\boldsymbol{e}_1 = \frac{1}{\sqrt{2}}(1,0,1)$ and $\boldsymbol{e}_3 = \frac{1}{\sqrt{6}}(1,2,-1)$. Hence, the minimum value 0 occurs at $\pm\frac{1}{\sqrt{2}}(1,0,1)$ and the maximum value 4 at $\pm\frac{1}{\sqrt{6}}(1,2,-1)$.

5. We see immediately that the columns of the matrix A are pairwise orthogonal and of length 1. Hence, A is orthogonal and, therefore, the matrix of an isometry. We have that

$$\det A = \frac{1}{27}\begin{vmatrix} 2 & -1 & 2 \\ 6 & 0 & 3 \\ 3 & 0 & 6 \end{vmatrix} = \frac{1}{27}\begin{vmatrix} 6 & 3 \\ 3 & 6 \end{vmatrix} = 1.$$

It follows that A is the matrix of a rotation. To find the axis of rotation, we solve the system $A\boldsymbol{x} = \boldsymbol{x}$.

$$\left[\begin{array}{ccc|c} -1 & -1 & 2 & 0 \\ 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \end{array}\right] \quad\Leftrightarrow\quad \left[\begin{array}{ccc|c} -1 & -1 & 2 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right] \quad\Leftrightarrow\quad \boldsymbol{x} = t(1,1,1).$$

The axis of rotation is therefore spanned by the vector $\boldsymbol{r} = (1, 1, 1)$. The vector $\boldsymbol{u} = (1, -1, 0)$ is orthogonal to $\boldsymbol{r}$ and mapped to

$$\frac{1}{3}\begin{bmatrix} 2 & -1 & 2 \\ 2 & 2 & -1 \\ -1 & 2 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

The angle θ between $\boldsymbol{u}$ and $\boldsymbol{v} = (1, 0, -1)$ is given by

$$\cos\theta = \frac{\boldsymbol{u}\cdot\boldsymbol{v}}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|} = \frac{1}{2}.$$

Hence, $\theta = \frac{\pi}{3}$. The determinant of the matrix having columns $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{r}$, in that order, is

$$\begin{vmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & -1 & 1 \end{vmatrix} > 0.$$

Therefore, the rotation appears anticlockwise on looking down from the tip of $\boldsymbol{r}$ towards the origin. To conclude, it is a rotation through the angle $\frac{\pi}{3}$ about the axis spanned by $\boldsymbol{r} = (1, 1, 1)$ and it appears anticlockwise on looking down from the tip of $\boldsymbol{r}$ towards the origin.

6. Suppose that

$$s_0\boldsymbol{x} + s_1A\boldsymbol{x} + s_2A^2\boldsymbol{x} + \cdots + s_{k-1}A^{k-1}\boldsymbol{x} + s_kA^k\boldsymbol{x} = \boldsymbol{0}.$$

Multiplying both sides by $I, A, A^2, \ldots, A^{k-1}, A^k$ and using that $A^i\boldsymbol{x} = \boldsymbol{0}$ for $i > k$, we get that

$$\begin{cases} s_0\boldsymbol{x} + s_1A\boldsymbol{x} + s_2A^2\boldsymbol{x} + \cdots + s_{k-1}A^{k-1}\boldsymbol{x} + \quad s_kA^k\boldsymbol{x} = \boldsymbol{0} \\ \qquad s_0A\boldsymbol{x} + s_1A^2\boldsymbol{x} + \cdots + s_{k-2}A^{k-1}\boldsymbol{x} + s_{k-1}A^k\boldsymbol{x} = \boldsymbol{0} \\ \qquad\qquad s_0A^2\boldsymbol{x} + \cdots + s_{k-3}A^{k-1}\boldsymbol{x} + s_{k-2}A^k\boldsymbol{x} = \boldsymbol{0} \\ \qquad\qquad\qquad \vdots \\ \qquad\qquad\qquad s_0A^{k-1}\boldsymbol{x} + \quad s_1A^k\boldsymbol{x} = \boldsymbol{0} \\ \qquad\qquad\qquad\qquad s_0A^k\boldsymbol{x} = \boldsymbol{0} \end{cases}.$$

Back-substituting and using that $A^i\boldsymbol{x} \neq \boldsymbol{0}$ for $i \leq k$, we now get that

$$s_0 = s_1 = s_2 = \cdots = s_{k-1} = s_k = 0,$$

which proves the statement.