LUND UNIVERSITY

Written Examination
Linear Algebra
Wednesday 29 October 2014
Duration: 8:00–13:00

Centre for Mathematical Sciences
Mathematics, Faculty of Science

Solutions

1. First we determine the kernel:

$$\begin{array}{rcl} x-y+z-w &=& 0 \\ 2x+3y-z &=& 0 \\ 4x+y+z-2w &=& 0 \end{array} \Leftrightarrow \begin{array}{rcl} x-y+z-w &=& 0 \\ 5y-3z+2w &=& 0 \end{array}$$

The solution is $(x,y,z,w) = (-2s+3t, 3s-2t, 5s, 5t)$, $s,t \in \mathbb{R}$. Thus the kernel is of dimension 2 and the vectors $(-2,3,5,0)$ and $(3,-2,0,5)$ are a basis for the kernel. By the dimension theorem, we have

$$dim(ker(F)) + dim(range(F)) = 4,$$

so the dimension of the range is 2. For example the first two columns of the matrix, the vectors $(1,2,4)$ and $(-1,3,1)$, belong to the range and they are linearly independent since they are not parallel. Then they are a basis for the range because any set of n linearly independent vectors in an n-dimensional space is a basis.
Answer: One correct answer is: The vectors $(-2,3,5,0)$ and $(3,-2,0,5)$ are a basis for the kernel. The vectors $(1,2,4)$ and $(-1,3,1)$ are a basis for the range. There are infinitely many other correct answers.

2. The points are on the line $y = at + b$ if

$$\begin{array}{rcl} -a+b &=& 2 \\ b &=& 2 \\ a+b &=& 1 \\ 2a+b &=& 0 \end{array} \Leftrightarrow A\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 1 \\ 0 \end{pmatrix} = Y$$

To obtain the best least squares fit we multiply the equation by A^T and get

$$A^TA\begin{pmatrix} a \\ b \end{pmatrix} = A^TY \Leftrightarrow \begin{pmatrix} 6 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}.$$

The solution is $a = -0.7$, $b = 1.6$.
Answer: The line with the best least squares fit to the points is $y = -0.7t + 1.6$.

3. The vectors $\mathbf{v_1}$ and $\mathbf{v_2}$ are linearly independent and they are a basis for the plane $V = span[\mathbf{v_1}, \mathbf{v_2}]$. We orthogonalize $\mathbf{v_1}$ and $\mathbf{v_2}$ to obtain an orthonormal basis for V:

$$\begin{array}{rcl} \mathbf{e_1} &=& \dfrac{\mathbf{v_1}}{||\mathbf{v_1}||} = \dfrac{1}{\sqrt{2}}(1,1,0) \\ \mathbf{f_2} &=& \mathbf{v_2} - (\mathbf{v_2}\cdot\mathbf{e_1})\mathbf{e_1} = (3,1,1) - \dfrac{4}{2}(1,1,0) = (1,-1,1) \\ \mathbf{e_2} &=& \dfrac{\mathbf{f_2}}{||\mathbf{f_2}||} = \dfrac{1}{\sqrt{3}}(1,-1,1) \end{array}$$

Please, turn over!

To extend $\mathbf{e_1}$ and $\mathbf{e_2}$ to an orthonormal basis for $\mathbb{R}^3$, we can take any vector that is orthogonal to $\mathbf{v_1}$ and $\mathbf{v_2}$, for example $(1,-1,2)$, and divide by the length. Thus we choose $\mathbf{e_3} = (1,-1,-2)/\sqrt{6}$

Answer: The vectors $\mathbf{e_1} = \frac{1}{\sqrt{2}}(1,1,0)$ and $\mathbf{e_2}\frac{1}{\sqrt{3}}(1,-1,1)$ are an orthonormal basis for V. With $\mathbf{e_3} = (1,-1,-2)/\sqrt{6}$, the vectors $\mathbf{e_1}$, $\mathbf{e_2}$ and $\mathbf{e_3}$ are a basis for $\mathbb{R}^3$. .

4. A matrix is a rotation if and only if it is orthogonal and the determinant is 1. The columns of the matrix A are orthogonal vectors with length 7 and the determinant is positive. Therefore we get an rotation matrix if we multiply A by $1/7$. To find the rotation axis we solve the system $cAX = X \iff AX = 7X$ which is equivalent to

$$\begin{aligned} -13x + 2y + 3z &= 0 \\ 2x - 10y + 6z &= 0 \\ 3x + 6y - 5z &= 0 \end{aligned} \Leftrightarrow \begin{aligned} x - 5y + 3z &= 0 \\ -63y + 42z &= 0 \\ 21y - 14z &= 0 \end{aligned}$$

which has the solution $t(1,2,3)$. To find the rotation angle, we choose any vector that is orthogonal to the rotation axis, for example $\mathbf{v} = (2,-1,0)$. Then $\frac{1}{7}A\mathbf{v} = -\mathbf{v}$ so the angle is π.

Answer: The matrix cA is a rotation if and only if $c = 1/7$ The rotation axis is $t(1,2,3)$ and the rotation angle is π.

5. The quadratic form is given by the symmetric matrix

$$A = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}.$$

The eigenvalues of A are the solutions of the characteristic equation

$$\begin{vmatrix} -\lambda & 2 & 2 \\ 2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = \begin{vmatrix} 4-\lambda & 4-\lambda & 4-\lambda \\ 2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = (4-\lambda)(4-\lambda)(-2-\lambda) = 0.$$

Thus the eigenvalues are $\lambda = 4$ (double) and $\lambda = -2$. This implies that we have

$$Q(x,y,z) = 4\hat{x}^2 + 4\hat{y}^2 - 2\hat{z}^2,$$

where $(\hat{x},\hat{y},\hat{z})$ are the coordinates of the vector (x,y,z) with respect to an ON-basis so

$$\hat{x}^2 + \hat{y}^2 + \hat{z}^2 = x^2 + y^2 + z^2.$$

It follows that

$$-2(x^2+y^2+z^2) = -2(\hat{x}^2+\hat{y}^2+\hat{z}^2) \le Q(x,y,z) \le 4(\hat{x}^2+\hat{y}^2+\hat{z}^2) = 4(x^2+y^2+z^2)$$

So the maximual value of Q when $x^2+y^2+z^2 = 1$ is 4 and the minimal value is -2.

Answer: Maximal value 4 and minimal value -2.

6. The rank of the matrix is 4 for all x such that $det(A) \neq 0$. We subtract the first row from all the following rows and get

$$det(A) = \begin{vmatrix} x & 1 & 1 & 1 \\ 1-x & x-1 & 0 & 0 \\ 1-x & 0 & x-1 & 0 \\ 1-x & 0 & 0 & x-1 \end{vmatrix}.$$

Now we can factor out $(x-1)$ from all the rows except the first and after that add the columns number two to four to the first one. This gives:

$$det(A) = (x-1)^3 \begin{vmatrix} x & 1 & 1 & 1 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{vmatrix} = (x-1)^3 \begin{vmatrix} x+3 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = (x-1)^3(x+3).$$

Thus the rank is 4 for all x except 1 and -3. When $x = 1$, all the columns of A are equal so the rank is 1. The rank of a matrix is not changed by column operations or row operations. So to find the rank of A when $x = -3$ we can check the last matrix in the calculation above with $x = -3$, namely

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

which clearly has 3 independent rows (and columns) and therefore rank equal to 3.

Answer: The rank is 1 for $x = 1$, 3 for $x = -3$ and 4 for all other values.