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MATEMATIK

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1.a) Characteristic equation for the homogeneous solution is

$$r^2 - 3r + 2 = 0, \qquad r_1 = 1, r_2 = 2.$$

The homogeneous solution is therefore

$$a_n^h = c_1 + c_2 2^n.$$

Ansatz for the particular solution (2 is a root of the characteristic equation):

$$a_n^p = cn2^n$$

gives $c = \frac{1}{2}$. Fitting the initial conditions, i.e. adapt the constants c_1, c_2 gives

$$a_n = 2 - 2^n + 2^{n-1}.$$

b) We apply the root crterion to $b_n = \left| \left(\frac{a_n}{a_{n+1}} \right)^n \right|$:

$$\lim_{n\to\infty} b_n^{1/n} = \lim_{n\to\infty} \left| \frac{2 - 2^n + n2^{n-1}}{2 - 2^{n+1} + (n+1)2^n} \right| = \lim_{n\to\infty} \left| \frac{\frac{1}{n2^{n-1}} - \frac{1}{n} + \frac{1}{2}}{\frac{1}{n2^{n-1}} - \frac{1}{n} + 1} \right| = \frac{1}{2} < 1.$$

Therefore the series converges.

2. We will use Euler's formula and set $w = e^{iz+i\frac{3}{2}\pi}$. We get

$$w + \frac{1}{w} = -4 \qquad \text{or} \qquad w^2 + 4w + 1 = 0.$$

Hence $w_{1,2} = -2 \pm \sqrt{3}$ och

$$i(z + \frac{3}{2}\pi) = \log(-2 \pm \sqrt{3}) = \ln(2 \mp \sqrt{3}) + (2k+1)\pi i, \qquad k \in \mathbb{Z}.$$

Finally,

$$z = -i\ln(2 \mp \sqrt{3}) + (2k - \frac{1}{2})\pi, \qquad k \in \mathbb{Z}.$$

b) The radius of convergence equals the distance from the origin (the series is centered at $z = 0$) to the closest singularity, i.e. to the closest solution of the equation from a). We have $\ln(2 + \sqrt{3}) = -\ln(2 - \sqrt{3})$ and therefore (with $k = 0$) the series converges for

$$z \in \left\{ w \in \mathbb{C} \,:\, |w| < \sqrt{\frac{\pi^2}{4} + \ln^2(2 + \sqrt{3})} \right\}.$$

3.a) f is even, so $b_k = 0$. For the exponential series we have

$$c_k = \frac{1}{2\pi} \in_{-\pi}^{\pi} |\sin t| e^{-ikt}\, dt = \frac{1}{\pi} \int_0^{\pi} \sin t e^{-ikt}\, dt.$$

Since $e^{-ik(t-\pi)} = (-1)^k e^{ikt}$ follows $c_k = 0$ for odd k.

$$c_{2k} = \frac{1}{\pi}\int_0^\pi \sin t e^{-2kit}\,dt = \frac{1}{2\pi i}\int_0^\pi \left(e^{it} - e^{-it}\right) e^{-2kit}\,dt = -\frac{1}{2\pi}\frac{4}{4k^2-1}$$
$$= -\frac{1}{2\pi}\frac{4}{(2k+1)(2k-1)}.$$

Using $a_k = c_k + c_{-k}$ (and the continuity of f) we get

$$|\sin t| = -\frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\frac{4}{(2k+1)(2k-1)}e^{2kit} = \frac{2}{\pi} - \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{4}{(2k+1)(2k-1)}\cos(2kt).$$

b) We will use Parseval's formula.

$$\frac{1}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi}|\sin t|^2\,dt = \frac{4}{\pi^2} + \frac{1}{2}\frac{16}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2(2k+1)^2}.$$

Therefore
$$\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2(2k+1)^2} = \frac{\pi^2}{16} - \frac{1}{2}.$$

4.a) u has to be harmonic, so

$$0 \equiv u''_{xx} + u''_{yy} = (g''(x) - g(x))\cos y \qquad \text{or} \qquad g''(x) = g(x).$$

The latter differential equation has the general solution $g(x) = c_1 e^x + c_2 e^{-x}$. We also have $f(0) = 1 + i \implies u(0,0) = 1 \implies g(0) = 1 \implies c_1 + c_2 = 1$. We seperate

$$u(x,y) = (c_1 e^x + (1-c_1)e^{-x})\cos y = c_1 e^x \cos y + (1-c_1)e^{-x}\cos y = u_1 + u_2.$$

The harmonic conjugates are $v_1(x,y) = c_1 e^x \sin y + d_1$ and $v_2(x,y) = -(1-c_1)e^{-x}\sin y + d_2$, $d_{1,2} \in \mathbb{R}$, respectively. Setting $z = x, y = 0$ and using the identity theorem we get

$$f(z) = c_1 e^z + (1-c_1)e^{-z} + id \qquad d \in \mathbb{R}.$$

Finally, $f(0) = 1 + i \implies d = 1$ and $f(i\frac{\pi}{2}) = 2i \implies c_1 = 1$. Hence, $f(z) = e^z + i$.

5.a) We use $\cos(2x) = \mathbb{R}e(e^{2ix})$ and consider

$$\int_C \frac{e^{2iz}}{1+z^2}\,dz = \int_{-R}^{R}\frac{e^{2ix}}{1+x^2}\,dx + \int_{C_R}\frac{e^{2iz}}{1+z^2}\,dz$$

where $C_R = \{z \in \mathbb{C} : |z| = R, \mathbb{I}m(z) > 0\}$. By Jordan's lemma the second integral on the right-hand-side vanishes as $R \to \infty$. Using the residue theorem ($z = i$ is the only singularity surrounded by C) we get

$$\int_{-\infty}^{\infty}\frac{\cos(2x)}{1+x^2}\,dx = \mathbb{R}e\left(\int_{-\infty}^{\infty}\frac{e^{2ix}}{1+x^2}\,dx\right) = \mathbb{R}e\left(2\pi i \cdot Res_{z=i}\left(\frac{e^{2iz}}{1+z^2}\right)\right) = \pi e^{-2}.$$

b) The function $f(z) = \dfrac{\cos(2z)}{1+z^2}$ has two singularities $z_{1,2} = \pm i$. Let γ be a closed smooth curve. By $W_i(\gamma)$, $W_{-i}(\gamma) \in \mathbb{Z}$ we denote the winding number (with orientation) of γ around i, respectively $-i$. That is we count how often the curve γ encircles each singularity taking into account the direction. The residue theorem says

$$\begin{aligned}\int_\gamma f(z)\,dz &= 2\pi i\,(W_i(\gamma)Res_{z=i}(f(z)) + W_{-i}(\gamma)Res_{z=-i}(f(z))) = W_i(\gamma)\pi\cos(2i) - W_{-i}(\gamma)\pi\cos(2i)\\ &= m\pi\cos(2i),\end{aligned}$$

where $m = m(\gamma) = W_i(\gamma) - W_{-i}(\gamma)$ is an arbitrary integer (depending on the curve γ).

6.a) Let $0 < r < 1$ and $|z| \le r$ then

$$\sum_{n=1}^{\infty}\left|\frac{z^n}{1-z^n}\right| \le \sum_{n=1}^{\infty}\frac{r^n}{1-r^n} \le \frac{1}{1-r}\sum_{n=1}^{\infty} r^n = \frac{r}{1-r} < \infty$$

and the series converges uniformly on $\{z \in \mathbb{C} : |z| \le r\}$ and hence converges for $|z| < 1$.
If $|z| > 1$ then

$$\lim_{n\to\infty}\left|\frac{z^n}{1-z^n}\right| \ge \lim_{n\to\infty}\frac{1}{1-\frac{1}{|z|^n}} = 1 > 0$$

and the series diverges since the terms do not tend to zero.
If $|z| = 1$ then $|1-z^n| \le 2$ and

$$\lim_{n\to\infty}\left|\frac{z^n}{1-z^n}\right| \ge \frac{1}{2} > 0$$

and again the series does not converge.
To summarize the series diverges for $|z| \ge 1$, converges for $|z| < 1$ (and converges uniformly on each disc $|z| \le r < 1$).
Remark: The series is not defined for $z = e^{i\frac{p}{q}\pi}$, $p, q \in \mathbb{Z}$.

b) Let $|z| \le r < 1$. By the view of a) the left-hand-side series converges uniformly and we can change the order of the terms in an arbitrary manner. By using $\dfrac{1}{1-z^k} = \sum_{l=0}^{\infty}(z^k)^l$ for $|z| < 1$, we conclude

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{z^n}{1-z^n} &= \sum_{k=1}^{\infty}\frac{z^k}{1-z^k} = \sum_{k=1}^{\infty} z^k\left(\sum_{l=0}^{\infty}(z^k)^l\right) = \sum_{k=1}^{\infty}\sum_{l=0}^{\infty} z^{(l+1)k}\\ &= \sum_{n=1}^{\infty}\sum_{k|n} z^n = \sum_{n=1}^{\infty}\psi(n)z^n.\end{aligned}$$