LUND UNIVERSITY

Centre for Mathematical Sciences
Mathematics, Faculty of Science

Written Examination
Fourier Analysis, 7.5 credits
Friday May 28, 2010
Time: 08.00-13.00

Use only the distributed paper sheets; write only on one side, and no more than one problem per sheet. Fill in the cover form fully and initialize each sheet. Write legibly. Give clear and brief arguments. Draw a picture if this helps.

1. Let

$$u(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & 1 \le x < \pi. \end{cases}$$

a) Find the cosine series of u.

b) Find the sum of the series

$$\sum_{k=1}^{\infty} \frac{\sin(2k)}{k}.$$

2. Find a solution u of the following problem

$$\begin{aligned} \partial_t u(x,t) &= \partial_x^2 u(x,t) && \text{when } t > 0 \text{ and } 0 < x < \pi, \\ \partial_x u(0,t) &= \partial_x u(\pi,t) = 0 && \text{when } t > 0, \\ u(x,0) &= \sin^2 x && \text{when } 0 < x < \pi. \end{aligned}$$

3. Find a function u such that

$$\int_{-\infty}^{\infty} \frac{u(x-y)}{1+y^2}\, dy = \frac{1}{4+x^2}.$$

4. Let $f(x) = xe^{-x}$ for $x > 0$ and $f(x) = 0$ for $x \le 0$.

a) Find the Fourier transform of f.

b) For each $\lambda > 0$, find the value of the integral

$$\int_{-\infty}^{\infty} \frac{\sin \lambda x}{x(1+ix)^2}\, dx.$$

5. Let $u(x) = x$ when $0 \le x \le \pi$.

a) Determine the numbers λ_n so that, for each positive integer N, the integral

$$\int_0^{\pi} |u(x) - \sum_1^N \lambda_n \sin(2nx)|^2\, dx$$

is as small as possible.

b) For these values of λ_n, find the limit of the integral in a) as $N \to \infty$.
Hint: One might want to consider the cosine series of the function $v(x) = x^2$, $0 \le x \le \pi$.