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MATEMATIK

SOLUTIONS
DISKRET MATEMATIK
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1. We can use inclusion/exclusion here. Let S all arrangements of the letters in SUMMER and use the conditions c_1= contains SUM, c_2= contains REM and c_3= contains MUM. Then $S_0 - S_1 + S_2 - S_3$ arrangements contain none of the substrings. $S_0 = \frac{6!}{2!}$, $S_1 = N(c_1) + N(c_2) + N(c_3) = 4! + 4! + 4!$, $S_2 = N(c_1c_2) + N(c_1c_3) + N(c_2c_3) = 2! + 0 + 2!$ and $S_3 = N(c_1c_2c_3) = 0$ (Note that SUM and MUM cannot both be substrings at the same time, and REM and MUM occur simultanously only if REMUM is a substring.) We find that $S_1 - S_2 + S_3 = \frac{6!}{2!} - 3 \cdot 4! + 2 \cdot 2! = 292$)

2. Applying Fermat's little theorem we find we can replace 2^74 by 1 and 5^61 by 5. Multiplying the second equation by $5^{-1} = 3$ we obtain the system

$$\begin{cases} x \equiv 1 \pmod 3 \\ x \equiv -1 \pmod 7 \end{cases}$$

By the Chinese Remainder Theorem (standard method for solution or trial and error) we find that $x = 13 + 21k$, $k \in \mathbb{Z}$

3. The characteristic equation $r^2 - r + 1 = 0$ has the roots $e^{\pm\frac{i\pi}{3}}$ so all complex solutions of the homogoneous equation are $Ce^{\frac{i\pi n}{3}} + De^{-\frac{i\pi n}{3}}$ for $C, D \in \mathbb{C}$. Rewriting using Euler's formula we get $a_n^(H) = A\cos(\frac{i\pi n}{3}) + B\sin(\frac{i\pi n}{3})$. The real solutions are obtained when $A, B \in \mathbb{R}$. Substitution of $a_n^(P) = E \cdot 5^n$ shows that $a_n^(P) = 2 \cdot 5^n$ and hence all real solutions are given by $a_n = A\cos(\frac{i\pi n}{3}) + B\sin(\frac{i\pi n}{3}) + 2 \cdot 5^n$ with $A, B \in \mathbb{R}$. Applying initial conditions we find the values $A = -3$ and $B = \sqrt{3}$ so $a_n = -3\cos(\frac{i\pi n}{3}) + \sqrt{3}\sin(\frac{i\pi n}{3}) + 2 \cdot 5^n$.

4. **a)** As any polynomial of degree one p is irreducible. Also q is irreducible being a degree three polynomial without zeroes. The factorisations $r(x) = (x+1)^2$ and $s(x) = (x-1)(x^2+x+1)$ show that r and s are reducible.

b) It is exactly those elements that have no common factor with $r(x) = x^2 + 2x + 1 = (x+1)^2$ that are invertible, that is $[x+2]$ and $[x-2]$. The divisions $r(x) = (x+2)x+1$ and $r(x) = (x-2)(x-1)-1$ show that $[x+2]^-1 = [-x]$ and $[x-2]^-1 = [x-1]$

5. Consider the binary linear code C with control matrix

$$H = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

a) H has $4 = n - m = 8 - m$ rows so $dim(C) = m = 4$.

b) Use theorem 3.10 from Andersson. No two columns of H are parallell (which in the binary case means equal). Column one, two and eight sum to zero and hence $d(C) = 3$.

c) The words 111111111, 010101010, 101010101, 110000010 have syndromes 1101, 1000, 0101, 0000. This shows that the last word is in the code, but none of the others. The first and third syndromes correspond to column number six and three in H. This means the first and third word has only one error and can be corrected to 11111011 and 10001010 respectively. The second syndrom has no unique coset leader. (001001 and 01000010 are two different coset elements of minimal weight.) This shows that the second word cannot be corrected.

6. **a)** We can use the generating function

$$(1+x+x^2+x^3)^4 = (\frac{1-x^4}{1-x})^4 = (1-4x^4+6x^8+\cdots)\sum_{j=0}^{\infty}\binom{-4}{j}(-x)^j$$

Our answer will be the coefficent of x^7 which equals $\binom{10}{3} - 4\binom{6}{3} = 120 - 80 = 40$

b) For each of the 40 cases in task a we can order the books in 7! ways so the answer is $7!40 = 201600$.

c) The cases can be counted using the exponential generating function

$$g(x) = (1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!})^4.$$

(The factorials in the denominators take care of the fact that we may permute within each shelf.) We search the coefficient of x^7, which can be found by straightforward multiplication or by using:

$$g(x) = (e^x - (\frac{x^4}{4!} + \frac{x^5}{5!} + \cdots))^4$$

Using the binomal theorem we only need to consider the two terms

$$(e^x)^4 - 4(e^x)^3(\frac{x^4}{4!} + \frac{x^5}{5!} + \cdots)$$

since everything else is degree at least eight. We get that the coefficent of x^7 is $\frac{4^7}{7!} - 4(\frac{3^3}{3!4!} + \frac{3^2}{2!5!} + \frac{3}{1!6!} + \frac{1}{7!}) = 7/3$ and the sought answer is $7!7/3 = 11760$.