

*No books, notes, computational devices, etc. are allowed. Use clear handwriting and give clear careful motivations. All answers should be fully simplified, but may contain factorials or powers. In particular they should not contain binomial coefficients or Stirling numbers. Fill in the form completely and write your personal identifier on each sheet of paper.*

1. How many strings can you make by rearranging the letters in SUMMER if the strings may not contain any of the substrings SUM, REM or MUM?
2. Find all integers satisfying the following system of congruences

$$\begin{cases} 2^{74}x \equiv 1 \pmod{3} \\ 5^{61}x \equiv 2 \pmod{7} \end{cases}$$

3. Find all real-valued solutions the recurrence relation  $a_{n+2} - a_{n+1} + a_n = 42 \cdot 5^n$ . Also find the solution among them that satisfies the initial conditions  $a_0 = -1$  and  $a_1 = 10$ .
4. a) Which of the polynomials  $p(x) = x + 2$ ,  $q(x) = x^2 + x + 1$ ,  $r(x) = x^2 + 2x + 1$  and  $s(x) = x^3 - 1$  are irreducible in  $\mathbb{Z}_5[x]$ ?  
b) Which of the elements  $[x + 1]$ ,  $[x + 2]$  and  $[x - 2]$  are invertible in the ring  $\mathbb{Z}_5[x]/(x^2 + 2x + 1)$ ? Compute the inverses of those elements that are invertible.
5. Consider the binary linear code  $C$  with control matrix

$$H = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

- a) What is the dimension of  $C$ ?
- b) What is the separation of  $C$ ?
- c) Determine which of the words

11111111, 01010101, 10101010, 11000001

are code words and correct the others if possible.

6. Suppose we have seven books and four different shelves. In how many ways can the books be placed on the shelves, with at most three books on the same shelf
  - a) if all books are identical?
  - b) if all books are distinct and we care about the ordering of the books within each shelf?
  - c) if all books are distinct, but we do not care about the ordering within each shelf?

**Good Luck!**