No books, notes, computational devices, etc. are allowed. Use only paper supplied by the department. Use clear handwriting and give clear careful motivations. All answers should be fully simplified. In particular they should not contain binomial coefficients, Stirling numbers or factorials. Fill in the form completely and write your personal identifier on each sheet of paper.

1. Solve the recurrence relation $x_{n+2} - 5x_{n+1} + 6x_n = 5 \cdot 2^n$ with initial conditions $x_0 = 5$ and $x_1 = 7$.

2. Consider the ring $R = \mathbb{Z}_5[x]/(x^3 + 2x^2 + 2x + 2)$.

   a) Compute the product $[x^2] \cdot [x^3]$ in R. (The answer should be on the form $[ax^2 + bx + c]$.)

   b) Find out if $[x^2 + 2x]$ is invertible in R and if so determine its inverse.

   c) Is R a field?

3.   a) Compute the Stirling numbers $S(n, k)$ for all integers n and k such that $1 \le k \le n \le 5$.

   b) In how many ways can we distribute 5 different balls between 3 identical boxes with no box left empty?

   c) How many surjective functions $f : \{A, B, C, D, E\} \to \{1, 2, 3\}$ are there?

4. Look at the linear code C over $\mathbb{Z}_3$ with generating matrix

$$G = \begin{pmatrix} 1 & 0 & 0 & 2 & 0 & 2 \\ 2 & 2 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 2 \end{pmatrix}$$

   a) How many words does C contain?

   b) Find a control matrix for C.

   c) What is the separation of C?

   d) For each of the words $w_1 = (111122)$, $w_2 = (220012)$ and $w_3 = (021101)$ decide if it is a code word or not. Also compute the corrected words for those words that are not in the code but are correctable.

5. You have a number of (identical) apples that you are going to distribute among five children: Alice, Bob, Clara, David and Eric. Clara, David and Eric each want either 5 or 11 apples so that they can make either a small or a large apple pie each. Bob is going to share his apples with his twin brother so he wants an even number of apples. How many distributions are there if you have

    a) 24 apples?

    b) 30 apples?

    c) k apples for any integer $k \geq 33$?

6. Assume that p and q are different primes. Look at the ring $\mathbb{Z}_{p^2 q}$ and the subset $\mathbb{Z}_{p^2 q}^*$ consisting of its invertible elements.

    a) Compute the size of the set $\mathbb{Z}_{p^2 q}^*$, that is the number of invertible elements in $\mathbb{Z}_{p^2 q}$.

    b) Is $\mathbb{Z}_{p^2 q}^*$ closed under multiplication?

    c) Is $\mathbb{Z}_{p^2 q}^*$ a subring of $\mathbb{Z}_{p^2 q}$?

    d) Let a be some fixed element of $\mathbb{Z}_{p^2 q}^*$. Define the map $\phi : \mathbb{Z}_{p^2 q}^* \to \mathbb{Z}_{p^2 q}^*$ by $\phi(x) = a \cdot x$. Show that ϕ is injective.

    e) Prove that an element x of $\mathbb{Z}_{p^2 q}$ is invertible if and only if $x^{p(p-1)(q-1)} = 1$.

Good Luck!