Suggested Solutions. Complex Analysis. Exam 2017-01-10

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Problem 1. By standard operations we see that

$$\kappa(z)=\sum_{k=0}^{\infty} kz^k=\frac{z}{(1-z)^2} \tag{1}$$

for $z\in\mathbb{D}$. It is evident from this formula that κ continues to an analytic function in $\Omega=\mathbb{C}\setminus\{1\}$. For $z\in\Omega$ we have that

$$\kappa(z)=\frac{1}{z-1}+\frac{1}{(z-1)^2},$$

which is the Laurent expansion for κ in Ω. □

Problem 2. Differentiating with respect to $\bar{z}$ we have that

$$\begin{aligned}\frac{\partial p}{\partial \bar{z}}&=6z^2\bar{z}^2-18z\bar{z}+12=6|z|^4-18|z|^2+12\\&=6(|z|^4-3|z|^2+2)=6(|z|^2-1)(|z|^2-2).\end{aligned}$$

By a standard result we have that the complex derivative $p'(z_0)$ exists if and only if $|z_0|=1$ or $|z_0|=\sqrt{2}$. □

Problem 3. By standard theory we have the Laurent expansion

$$f(z)=\sum_{n=-1}^{\infty} a_n(z-a)^n$$

for z in some punctured neighborhood of a, where a_{-1} is the residue for f at a. Now

$$g(z)=(z-a)^m f(z)=a_{-1}(z-a)^{m-1}+\dots$$

for z in some neighborhood of a. Observe that a_{-1} is the coefficient for $(z-a)^{m-1}$ in the power series expansion for g around a. By a well-known coefficient formula we have that

$$g^{(m-1)}(a)/(m-1)!=a_{-1},$$

which proves the residue formula in the problem. □

Remark. The point of Problem 3 is that the residue formula considered remains true if we overestimate the order of the pole.

Problem 4. Since the integrand is even we have that

$$I=\int_0^{\pi}\frac{\cos(\theta)}{2+\cos(\theta)}\,d\theta=\frac{1}{2}\int_{-\pi}^{\pi}\frac{\cos(\theta)}{2+\cos(\theta)}\,d\theta.$$

Also by a division argument and periodicity we have that

$$I=\pi-\int_0^{2\pi}\frac{d\theta}{2+\cos(\theta)}.$$

From Gamelin Section VII.3 we know that

$$\int_0^{2\pi}\frac{d\theta}{a+\cos(\theta)}=\frac{2\pi}{\sqrt{a^2-1}}$$

for $a > 1$. We conclude that

$$I = \pi - \frac{2\pi}{\sqrt{3}} = \Big(1 - \frac{2}{\sqrt{3}}\Big)\pi.$$

□

Problem 5. Recall that for $\alpha \in \mathbb{D}$ the function

$$\varphi_\alpha(z) = \frac{\alpha - z}{1 - \bar{\alpha} z}$$

maps $\mathbb{D}$ one-to-one onto itself. Also $\varphi_\alpha(0) = \alpha$ and $\varphi_\alpha \circ \varphi_\alpha = \mathrm{id}$, where id is the identity map.

Assume now that $\varphi \in \mathrm{Aut}(\mathbb{D})$ has a fixed point $\alpha \in \mathbb{D}$. Consider the function

$$h = \varphi_\alpha \circ \varphi \circ \varphi_\alpha$$

in $\mathrm{Aut}(\mathbb{D})$. Observe that h fixes the origin, that is, $h(0) = 0$. An application of Schwarz lemma gives that

$$h(z) = cz, \quad z \in \mathbb{D}, \tag{2}$$

for some unimodular constant c. Assume now that φ has also another fixed point $\beta \in \mathbb{D}$. Substituting $\varphi_\alpha(\beta)$ for z in (2) we get

$$\varphi_\alpha(\beta) = c\varphi_\alpha(\beta),$$

so that $c = 1$ by cancellation since $\varphi_\alpha(\beta) \neq 0$. Solving for φ we see that $\varphi = \mathrm{id}$. □

Remark. A well-known result says that if a Möbius transformation has three (3) distinct fixed points in the extended complex plane $\mathbb{C}_\infty$, then it equals the identity map. Problem 5 says that within the subgroup $\mathrm{Aut}(\mathbb{D})$ of the Möbius group $\mathcal{M} = \mathrm{Aut}(\mathbb{C}_\infty)$ it suffices with two (2) distinct fixed points in $\mathbb{D}$ to characterize the identity map.

Problem 6. Observe that

$$\cos^2(\theta) = \Big(\frac{e^{i\theta} + e^{-i\theta}}{2}\Big)^2 = \frac{1}{4}e^{-i2\theta} + \frac{1}{2} + \frac{1}{4}e^{i2\theta}$$

for $e^{i\theta} \in \mathbb{T}$. From this formula we see that

$$u(z) = \frac{1}{4}\bar{z}^2 + \frac{1}{2} + \frac{1}{4}z^2, \quad z \in \mathbb{C},$$

is an entire harmonic function such that $u(e^{i\theta}) = \cos^2(\theta)$ for $e^{i\theta} \in \mathbb{T}$. □