Suggested Solutions. Complex Analysis. Exam 2016-12-17

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Problem 1. From the course we know that

$$g(z) = \sum_{k=0}^{\infty} (k+1) z^k = \frac{1}{(1-z)^2} \tag{1}$$

for $z \in \mathbb{D}$. It is evident from this formula that g continues to an analytic function in $\mathbb{C} \setminus \{1\}$. For $|z| > 1$ we have that

$$g(z) = \frac{1}{z^2} \frac{1}{(1-1/z)^2} = \frac{1}{z^2} \sum_{k=0}^{\infty} (k+1) \frac{1}{z^k} = - \sum_{m=-\infty}^{-2} (m+1) z^m,$$

where the mid equality follows by (1). □

Problem 2. The function

$$g(z) = 2z + 1 - 3i$$

maps $\mathbb{D}$ one-to-one onto D but does not satisfy the desired normalization conditions. We shall modify g correspondingly.

Recall that for $\alpha \in \mathbb{D}$ the function

$$\varphi_\alpha(z) = \frac{\alpha - z}{1 - \bar{\alpha} z}$$

maps $\mathbb{D}$ one-to-one onto itself. Also $\varphi_\alpha(0) = \alpha$. The derivative of φ_α is given by

$$\varphi_\alpha'(z) = -\frac{1 - |\alpha|^2}{(1 - \bar{\alpha} z)^2}.$$

Notice that $g((1+i)/2) = 2 - 2i$. The function

$$h = g \circ \varphi_{(1+i)/2}$$

maps $\mathbb{D}$ one-to-one onto D and also satisfies that $h(0) = 2 - 2i$. The same is true for each of the functions

$$f_{e^{i\theta}}(z) = g \circ \varphi_{(1+i)/2}(e^{i\theta} z),$$

where $e^{i\theta} \in \mathbb{T} = \partial\mathbb{D}$ (we compose with a rotation of the disc).

We shall next determine $e^{i\theta} \in \mathbb{T}$ such that $f_{e^{i\theta}}'(0) > 0$. Differentiating we have that

$$f_{e^{i\theta}}'(0) = g'((1+i)/2) \varphi_{(1+i)/2}'(0) e^{i\theta} = -e^{i\theta}.$$

We are now led to set

$$f(z) = f_{-1}(z) = \frac{4 - 4i + (2 - 4i) z}{2 + (1 - i) z}.$$

This function f maps $\mathbb{D}$ one-to-one onto D and satisfies that $f(0) = 2 - 2i$ and $f'(0) = 1 > 0$. □

Problem 3. Differentiating on $\bar{z}$ we have that

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial u}{\partial \bar{z}} + i \frac{\partial v}{\partial \bar{z}} = \frac{1}{2}\Big(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \Big) + \frac{i}{2} \Big(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \Big) = 0$$

in Ω by the Cauchy-Riemann equations. This proves that f is analytic in Ω. □

Problem 4. By standard theory we have the Laurent expansion

$$f(z) = \sum_{n=-1}^{\infty} a_n (z-a)^n$$

in some punctured neighborhood of a, where a_{-1} is the residue for f at a. Now

$$g(z) = (z-a)^2 f(z) = a_{-1}(z-a) + \dots$$

for z some neighborhood of a. Observe that a_{-1} is the coefficient for $(z-a)$ in the power series expansion of g around a. By a well-known coefficient formula we have that $g'(a) = a_{-1}$, which proves the residue formula in the problem. □

Remark. The point of Problem 4 is that the residue formula considered remains true if we overestimate the order of the pole.

Problem 5. Set

$$I = \int_0^{\infty} \frac{\log x}{(x+1)^2}\, dx.$$

By the change of variables $x \mapsto 1/x$ we have that

$$I = -\int_0^{\infty} \frac{\log x}{(x+1)^2}\, dx = -I.$$

Solving for I we conclude that $I = 0$. □

Problem 6. Consider the help function

$$h(z) = \frac{1}{f(1/z)}.$$

The function h is analytic in a punctured neighborhood of the origin. By assumption we have that $h(z) \to 0$ as $z \to 0$. A well-known result now gives that the singularity of h at the origin is removable (see Gamelin Section VI.2). Denote by $N \geq 0$ the order of the zero of h at the origin. Solving for f we have that

$$f(z) = \frac{1}{h(1/z)} = O(|z|^N)$$

as $|z| \to \infty$. A Liouville theorem now gives that f is a polynomial of degree at most N (see Gamelin Section IV.5; Exercise IV.5:4). □

Remark. A byproduct of Problem 6 is a description of the automorphism group $\mathrm{Aut}(\mathbb{C})$ of the complex plane $\mathbb{C}$. An analytic function φ maps $\mathbb{C}$ one-to-one onto itself if and only if it has the form

$$\varphi(z) = az + b, \quad z \in \mathbb{C},$$

for some complex numbers $a \neq 0$ and b.