LUNDS UNIVERSITET

Tentamensskrivning
Analytiska Funktioner
Onsdag den 17 december 2014
Skrivtid: 8.00–13.00

Matematikcentrum
Matematik NF

Inga hjälpmedel. Använd institutionens papper, skriv på bara den ena sidan och högst en uppgift på varje papper. Skriv tydligt, ge klara och kortfattade motiveringar, rita gärna figur i förekommande fall och ge tydliga svar. Fyll i omslaget fullständigt och skriv initialer på varje papper.

No books, notes, computational devices etc. are allowed. Use paper supplied by the Department, write only on one side of each paper, and treat at most one excercise on each paper. Use clear handwriting and give clear careful motivations. Fill in the form completely and write your name on each sheet of paper.

1. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk and let $\mathbb{C}^+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ be the upper half plane.

Find a conformal map $\phi : \mathbb{C}^+ \to \mathbb{D}$ with $\phi(i) = 1/2$ and $i\phi'(i) \in \mathbb{R}$, $i\phi'(i) < 0$. Show that this map is unique.

Solution.

Let $f : \mathbb{C}^+ \to \mathbb{D}$ be the Cayley map, $F(z) = \frac{z-i}{z+i}$. Let ψ be the Möbius transform interchanging 0 and $\frac{1}{2}$, $\psi(z) = \frac{1/2-z}{1-z1/2}$. Then

$$\phi : \mathbb{C}^+ \to \mathbb{D}, \qquad \phi(z) = \psi(F(z))$$

is a conformal map, as composition of conformal maps $\mathbb{C}^+ \to \mathbb{D}$ and $\mathbb{D} \to \mathbb{D}$. We have

$$\phi(i) = \psi(0) = \frac{1}{2}$$

and

$$i\phi'(i) = iF'(i)\psi'(F(i)) = i\frac{2i}{(2i)^2}\frac{-1-(-1/2)(1/2)}{1} = i\frac{1}{2i}(-1+1/4) < 0.$$

Concerning uniqueness, $\phi : \mathbb{C}^+ \to \mathbb{D}$ with $\phi(i) = 1/2$ and $i\phi'(i) \in \mathbb{R}, i\phi'(i) < 0$ is equivalent to $(-i\phi) : \mathbb{C}^+ \to \mathbb{D}$ with $(-i\phi)(i) = -1/2i$ and $(-i\phi)'(i) \in \mathbb{R}, (-i\phi)'(i) > 0$, and the latter map is unique by the Riemann mapping theorem. Thus ϕ is the unique conformal map with these properties.

2. Consider the function $f : \mathbb{C} \to \mathbb{C}$,

$$f(z) = \sin(\bar{z}z) + \cos(\bar{z}z).$$

Determine at which points $z_0 \in \mathbb{C}$ the function is holomorphic; that means, the complex derivative

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

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exists.

Solution. The function f has continuous partial derivatives. Therefore, we only have to check at which points z_0 the Cauchy-Riemann equations hold, or equivalently, that

$$\frac{\partial f}{\partial \bar{z}}(z_0) = 0.$$

Clearly

$$\frac{\partial f}{\partial \bar{z}}(z_0) = z_0 \cos(\bar{z}_0 z_0) - z_0 \sin(\bar{z}_0 z_0) = z_0(\cos(|z_0|^2) - \sin(|z_0|^2))$$

so for the expression to be 0, either $z_0 = 0$, or

$$\sin(|z_0|^2) = \cos(|z_0|^2) \Leftrightarrow |z_0|^2 = \pi/4 + n\pi, n \in \mathbb{N}_0 \Leftrightarrow |z_0| = (\pi/4 + n\pi)^{1/2}, n \in \mathbb{N}_0.$$

3. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function with $|f(z)| \leq 1 + |z|^{5/2}$ for all $z \in \mathbb{C}$. Show that f is a polynomial of degree at most 2.

Solution. Since f is entire, it can be written as a power series centered at 0, which converges normally on $\mathbb{C}$, so

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0) z^n$$

By the Cauchy Inequalities,

$$|f^{(n)}(0)| \leq \frac{n!}{R^n} \sup_{|z|=R} |f(z)| \leq \frac{n!}{R^n}(|R|^{5/2} + 1) \overset{R\to\infty}{\to} 0$$

for $n \geq 3$. Hence the power series above consists only of the first three terms, and f is a polynomial of degree at most 2.

4. For each $n \in \mathbb{N}$, determine the number of zeros of the function $p(z) = 4z^n - \sin z$ in the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

Solution: By Rouché's Theorem, two analytic functions f and $f + g$ have the same number of zeroes inside the piecewise smooth closed simple curve γ, provided that $|f(z)| > |g(z)|$ for all $z \in \gamma$. Let $f(z) = 4z^n$ and $g(z) = -\sin z$. Then for $|z| = 1$, $z = x + iy$,

$$|g(z)| = |\sin z| = \frac{1}{2}|e^{iz} - e^{-iz}| \leq \frac{1}{2}(e^{-y} + e^{y}) \leq e < 4 = |f(z)|.$$

Hence by Rouché's Theorem, p and f have the same umber of zeros on $\mathbb{D}$, namely n.

5. Consider the function f given by

$$f(z) = \frac{1}{\frac{1}{2} - z} + \operatorname{Log}(z + 1) \qquad (z \in \mathbb{D}, z \neq \frac{1}{2}),$$

where Log denotes the principal branch of the logarithm and $\mathbb{D}$ denotes the unit disk, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Find the Laurent series expansion of f around 0 which converges in $z = 3/4$. For both the principal and the holomorphic part of the Laurent series expansion, determine the domain of convergence.

Solution. The function $z \mapsto \text{Log}\,(1+z)$ has the following Taylor series expansion around 0:

$$\text{Log}\,(1+z) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{z^n}{n}.$$

The radius of convergence is 1, since Log has a singularity in 0.
The function $z \mapsto \frac{1}{1/2-z}$ has the following Taylor series expansion around 0, which converges for $0 \leq |z| < 1/2$:

$$\frac{1}{1/2-z} = 2\frac{1}{1-2z} = \sum_{n=0}^{\infty} 2^n z^n$$

and the following Laurent series expansion around 0, which converges for $|z| > 1/2$:

$$\frac{1}{1/2-z} = \frac{1}{z}\frac{1}{1/(2z)-1} = -\frac{1}{z}\sum_{n=0}^{\infty} 2^{-n} z^{-n} = -\sum_{n=0}^{\infty} 2^{-n} z^{-(n+1)}$$
$$= -\sum_{n=1}^{\infty} 2^{-n+1} z^{-n} = -\sum_{n=-\infty}^{-1} 2^{n+1} z^n$$

Thus the Laurent series expansion of f around 0 which converges in $z = 3/4$ is

$$f(z) = -\sum_{n=-\infty}^{-1} 2^{n+1} z^n + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{z^n}{n}.$$

Its principal part has domain of convergence $\{z \in \mathbb{C} : |z| > 1/2\}$, its holomorphic part has domain of convergence $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

6. Compute the integral

$$\int_0^{\infty} \frac{x^2}{x^4+1} dx.$$

Solution. We use the Residue theorem.
Observe that $f(z) = \frac{z^2}{z^4+1}$ has 2 simple poles in the upper half plane, namely at $z_1 = e^{\pi i/4}$ and $z_2 = e^{3\pi i/4}$. The residue of f in z_1 is

$$\text{res}_{z_1} f = \lim_{z \to z_1} (z - z_1) \frac{z^2}{x^4+1} = \frac{z_1^2}{4z_1^3} = \frac{1}{4}\frac{1}{z_1} = \frac{1}{4}e^{-\pi i/4} = \frac{1}{4}\frac{1-i}{\sqrt{2}}$$

The residue of f in z_2 is

$$\text{res}_{z_2} f = \lim_{z \to z_2} (z - z_2) \frac{z^2}{z^4+1} = \frac{z_z^2}{4z_2^3} = \frac{1}{4}\frac{1}{z_2} = \frac{1}{4}e^{-3\pi i/4} = \frac{1}{4}\frac{-1-i}{\sqrt{2}}.$$

Let γ_R denote the closed path consisting of the interval $[-R, R]$ and the semicircle S_R in the upper half-plane with radius R in positive direction. Then by the Residue Theorem,

$$\int_{\gamma_R} f(z)dz = 2\pi i(\text{res}_{z_1} f + \text{res} z_2 f) = \frac{1}{\sqrt{2}}\pi.$$

Furthermore,

$$\lim_{R\to\infty} |\int_{S_R} f(z)dz| \leq \lim_{R\to\infty} \pi R \frac{R^2}{R^4-1} = 0.$$

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Hence

$$\begin{aligned}\int_0^\infty \frac{x^2}{1+x^4}dx &= \frac{1}{2}\int_{-\infty}^\infty \frac{x^2}{1+x^4}dx \\ &= \frac{1}{2}\lim_{R\to\infty}\int_{-R}^{R}\frac{x^2}{1+x^4}dx \\ &= \frac{1}{2}\lim_{R\to\infty}\int_{\gamma_R} f(z)dz - \int_{S_R} f(z)dz \\ &= \frac{1}{2\sqrt{2}}\pi.\end{aligned}$$