Exam Analytic functions 11 Dec. 2013

Allowed tools are pen and rubber. Remember to motivate all conclusions, and to represent your solution in an organized and easily readable fashion.

Problem 1. Find all analytic functions f such that $|f(z)| \leqslant |z^5 + 3z - 1|$. (5p)

Solution: Let f be a solution to the above problem and let Λ be the zeroes of $z^5 + 3z - 1$. Λ has at most 5 elements, by the fundamental theorem of algebra. Consider the function $g(z) = f(z)/(z^5+3z-1)$ defined on $\mathbb{C}\backslash\Lambda$. It is clearly analytic and bounded by 1. Therefore its poles are removable, by Riemann's theorem on removable singularities. We can thus extend the definition of g to all of $\mathbb{C}$ by continuity, and the resulting function is analytic as well, i.e. it is entire. Liouville's theorem now yields that $g(z)$ is actually a constant $c \in \mathbb{C}$ with $|c| \leqslant 1$. Summing up, if f is a solution, then it has to be of the form $f(z) = c(z^5+3z-1)$ with $|c| \leqslant 1$. Since it is immediate that all such functions indeed solve the original problem, this is the answer.

Problem 2. Let $f(z) = \frac{e^z}{\cos z}$ and let C denote the circle $\{z \in \mathbb{C} : \ |z| = 2\}$.

i) Compute $\int_C f(z)\ dz$, where C is positively oriented. (4p)

ii) Compute $\int_C (f(z))^2\ dz$, where C is negatively oriented. (1p)

Solution: i) $\cos z$ has only zeroes at $\pm\pi/2$ inside $\mathbb{C}$. Since these are simple and since $\frac{d}{dz}\cos(z) = -\sin(z)$, we conclude that

$$res(f, \pi/2) = \mathrm{e}^{\pi/2}/(-1)$$

and

$$res(f, \pi/2) = \mathrm{e}^{-\pi/2}/(1).$$

The residue theorem thus implies that

$$\int_C f(z)\ dz = 2\pi\mathrm{i}(-\mathrm{e}^{\pi/2} + \mathrm{e}^{-\pi/2}).$$

ii) The fact that C is negatively oriented only changes the sign, i.e. the residue formula now reads

$$\int_C (f(z))^2\ dz = -2\pi\mathrm{i}\Big(res(f^2, \pi/2) + res(f^2, -\pi/2)\Big).$$

Since we now have second order poles, the determination of the residues is more complicated. One option is to use the formula

$$res(f^2, \pi/2) = \lim_{z\to\pi/2} \frac{d}{dz}(z - \frac{\pi}{2})^2 f^2(z)$$

and a corresponding formula for $-\pi/2$, but the limits gets messy to compute. Another option is to compute the first terms in the Laurent series:

$$\frac{\mathrm{e}^{\frac{\pi}{2}+z}}{\cos(\frac{\pi}{2}+z)} = \mathrm{e}^{\frac{\pi}{2}}\frac{\mathrm{e}^z}{-\sin(z)} = -\mathrm{e}^{\frac{\pi}{2}}\mathrm{e}^z z^{-1}(1 - \frac{z^2}{6} + O(z^4))^{-1} =$$

$$= -\mathrm{e}^{\frac{\pi}{2}}(1 + z + O(z^2))z^{-1}(1 + \frac{z^2}{6} + O(z^4))$$

where we have assumed that z is small enough to use the formula $1/(1-x) = \sum_{k=0}^{\infty} x^k$. Hence we get

$$f(\frac{\pi}{2}+z) = -\mathrm{e}^{\pi/2}\left(\frac{1}{z}+1+O(z)\right)$$

for z near 0, and so

$$(f(\frac{\pi}{2}+z))^2 = \mathrm{e}^{\pi}\left(\frac{1}{z^2}+\frac{2}{z}+O(1)\right).$$

Summing up, $res(f^2,\pi/2) = 2\mathrm{e}^{\pi}$. A similar calculation yields $res(f^2,\pi/2) = 2\mathrm{e}^{-\pi}$, so

$$\int_C (f(z))^2\, dz = -4\pi\mathrm{i}(\mathrm{e}^{\pi}+\mathrm{e}^{-\pi}).$$

Problem 3. Let $\Omega = \{z \in \mathbb{C}\backslash\{0\} : \ |\arg z| < \pi/6\}$.

i) Find a holomorphic bijection between Ω and the unit disc $\mathbb{D}$. (3p)
ii) Find *the* holomorphic bijection ϕ between Ω and the unit disc $\mathbb{D}$ such that $\phi(2) = 0$ and $\phi'(2) > 0$. (1p)
iii) Motivate theoretically the use of the word *the* in ii). (1p)

Solution: i) z^3 is a bijection with the right half plane, so iz^3 is bijective with $\mathbb{C}^+$. Hence

$$\frac{\mathrm{i}z^3-\mathrm{i}}{\mathrm{i}z^3+\mathrm{i}} = \frac{z^3-1}{z^3+1}$$

is one correct answer.

ii) $\mathrm{i}z^3/8$ is another bijection from Ω to $\mathbb{C}^+$ that sends 2 to i. Since $\frac{z-\mathrm{i}}{z+\mathrm{i}}$ sends i to 0, the sought function is easily seen to be

$$\phi(z) = \frac{z^3-8}{z^3+8}.$$

iii) The found map is unique by the Riemann mapping theorem.

Problem 4. Consider the function

$$f(z) = z^9 + 10z^3 + \mathrm{e}^z.$$

- How many zeroes does it have inside the disc $\{z \in \mathbb{C} : \ |z| < 3\}$? (3p)
- How many zeroes does it have inside the annulus $\{z \in \mathbb{C} : \ 1 < |z| < 3\}$? (2p)

Solution: i) On $|z| = 3$, $|z^9| = 3^9$ which is much bigger than $|10z^3 + \mathrm{e}^z|$, which is seen e.g. by the estimate

$$|10z^3 + \mathrm{e}^z| \leqslant 3^3 \cdot 3^3 + e^3 \leqslant 3^6 + 3^3 \leqslant 3^7.$$

Rouche's theorem thus says that f has 9 zeroes in $\{z \in \mathbb{C} : \ |z| < 3\}$.

ii) On $|z| = 1$, $|10z^3| = 10$ whereas $|z^9 + \mathrm{e}^z| \leqslant 1 + 3 = 4$. This implies that f has no zeroes on $\{z : |z| = 1\}$. As above we conclude that it has 3 zeroes inside. Thus f has 6 zeroes in the given annulus.

Problem 5. Let $t(x)$ be the "tent-function" defined by $\max(1-x,0)$ on $\mathbb{R}^+$ and by the requirement that it is symmetric. Its Fourier transform is given by

$$\hat{t}(\xi) = \begin{cases} \frac{1}{2\pi^2}\frac{1-a\cos(b\xi)}{\xi^2}, & \xi \neq 0 \\ c, & \xi = 0 \end{cases}$$

where a, b and c are positive real numbers that one can compute by direct integration. However, there are other ways, and this exercise revolves around this.

i) Compute a, b and c by direct integration. (1p) *Hint: After evaluating the integral on* $[0,1]$*, the other integral can be obtained easily by a change of variables.*

ii) Is the singularity of $\hat{t}$ at 0 removable, pole or essential? Give a theoretical motivation and based on this, determine a. (1p)

iii) The value of b can be obtained by an argument based on the Paley-Wiener theorem. Fill in the details. (2p)

iv) With a and b at hand, determine c by a theoretical argument. (1p)

v) Compute the Hadamard factorization of $\hat{t}$ and shorten the expression as far as possible. (2p)

Solution: i) By partial integration we get for $\xi \neq 0$ that

$$\int_0^1 (1-x)\mathrm{e}^{-2\pi \mathrm{i} x\xi} = \frac{1}{2\pi\mathrm{i}\xi} + \frac{1-\mathrm{e}^{-2\pi\mathrm{i}\xi}}{4\pi^2\xi^2}.$$

The change of variables $x \to -x$ gives

$$\int_{-1}^0 (1+x)\mathrm{e}^{-2\pi\mathrm{i}x\xi} = \int_0^1 (1-x)\mathrm{e}^{-2\pi\mathrm{i}x(-\xi)} = \frac{1}{2\pi\mathrm{i}(-\xi)} + \frac{1-\mathrm{e}^{-2\pi\mathrm{i}(-\xi)}}{4\pi^2(-\xi)^2}.$$

Adding up gives

$$\hat{t}(\xi) = \frac{1-\cos(2\pi\xi)}{2\pi^2\xi^2}.$$

Thus $a=1$ and $b=2\pi$. Finally, $\hat{t}(0)$ is the integral of t which is 1, so $c=1$.

ii) The Fourier transform of a function with compact support is entire, so the singularity is certainly removable. Since $\cos(0)=1$, this is only possible if $a=1$. In the remainder we treat ξ as the real part of ζ and consider $\hat{t}$ to be defined on all of $\mathbb{C}$.

iii) Since t is supported on $[-1,1]$, Paley-Wiener says that $|\hat{t}(\zeta)| \leqslant A\mathrm{e}^{2\pi|\zeta|}$ for some $A>0$. This is only possible if $b \leqslant 2\pi$, (due to the fact that $|\cos(b\zeta)|$ behaves like $\mathrm{e}^{b|\zeta|}/2$ for large values of ζ on the imaginary axis, and the other factors in $\hat{t}$ are negligible). In a similar fashion, it is easy to see that we have $|\hat{t}(\zeta)| \leqslant A\mathrm{e}^{b|\zeta|}$ for some $A>0$, by basic estimates of the explicit expression for $\hat{t}$. Thus, if $b<2\pi$, the converse part of Paley-Wiener says that t is supported on the smaller interval $[-b/2\pi, b/2\pi]$, which is clearly a contradiction. Hence $b=2\pi$.

iv) Since $\hat{t}(\zeta)$ is entire, it is continuous at $\zeta=0$. The series expansion is

$$\hat{t}(\zeta) = \frac{1}{2\pi^2}\frac{1-\cos(2\pi\zeta)}{\zeta^2} = \frac{1}{2\pi^2}\frac{1-(1-\frac{(2\pi\zeta)^2}{2}+O(\zeta^4))}{\zeta^2} = 1+O(\zeta^2).$$

The continuity thus forces $c=1$.

v) The zeroes of $\hat{t}$ have order 2 and are located at $\mathbb{Z}\backslash\{0\}$. Since $\hat{t}$ is the Fourier transform of a function with compact support, it has exponential type. Thus Hadamard says that

$$\hat{t}(\zeta) = \mathrm{e}^{A+B\zeta} \prod_{n\in\mathbb{Z}\backslash\{0\}} \left(\left(1 - \frac{\zeta}{n}\right) \mathrm{e}^{\zeta/n} \right)^2 = \mathrm{e}^{A+B\zeta} \prod_{n=1}^{\infty} \left(1 - \frac{\zeta}{n}\right)^2,$$

where A and B are constants. Since $\hat{t}(0) = 1$, we have $A = 0$. Since both $\hat{t}$ and the infinite product are even functions on $\mathbb{R}$, we must have $B = 0$ as well. Thus

$$\hat{t}(\zeta) = \prod_{n=1}^{\infty} \left(1 - \frac{\zeta}{n}\right)^2.$$