LUNDS UNIVERSITET

Matematikcentrum
Matematik NF

Tentamensskrivning
Analytiska Funktioner
Fredag den 14 december 2012
Skrivtid: 10.15–15.15

Inga hjälpmedel. Använd institutionens papper, skriv på bara den ena sidan och högst en uppgift på varje papper. Skriv tydligt, ge klara och kortfattade motiveringar, rita gärna figur i förekommande fall och ge tydliga svar. Fyll i omslaget fullständigt och skriv initialer på varje papper.

No books, notes, computational devices etc. are allowed. Use paper supplied by the Department, write only on one side of each paper, and treat at most one excercise on each paper. Use clear handwriting and give clear careful motivations. Fill in the form completely and write your name on each sheet of paper.

Note that partial solutions, solution attempts etc. will be given attention in the evaluation and can amount to (partial) points in the scoring. Please do not hesitate to give such documentation.

1. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk, let $R > 0$, $c \in \mathbb{C}$, and let $D_R(c) = \{z \in \mathbb{C} : |z - c| < R\}$ be the disk with radius R and centre c .

1. Find a biholomorphic map
$$f : \mathbb{D} \to D_R(c)$$
such that
$$f(0) = c, \quad f'(0) > 0.$$
Show that f is the unique biholomorphic map with these properties.

2. Now let $w_0 \in D_R(c)$. Find a biholomorphic map
$$g : \mathbb{D} \to D_R(c)$$
such that
$$g(0) = w_0, \quad g'(0) < 0.$$
Notice the change of signs!

Solution.

1. Let $f : \mathbb{D} \to D_R(c)$, $f(z) = Rz + c$. Then f is biholomorphic, and $f(0) = c$, $f'(0) = R > 0$ as required. Moreover, if $h : \mathbb{D} \to D_R(c)$ is any biholomorphic map with these properties, then
$$h^{-1} \circ f : \mathbb{D} \to \mathbb{D}$$
is an automorphism of the unit disk with $h^{-1} \circ f(0) = 0$, and therefore a rotation. Since $(h^{-1} \circ f)'(0) = f'(0)\frac{1}{h'(0)} > 0$, $h^{-1} \circ f = id_{\mathbb{D}}$, and therefore $h = f$.

2. Let $\alpha = f^{-1}(w_0) = \frac{1}{R}(w_0 - c)$ and
$$g = f \circ \psi_\alpha,$$

Var god vänd!

where ψ_α is the Möbius transformation interchanging 0 and α. Then

$$g(0) = f(\alpha) = w_0$$

and

$$g'(0) = f'(\alpha)\psi'_\alpha(0) = R(-1 + |\alpha|^2) < 0.$$

2. Consider the function $f : \mathbb{C} \to \mathbb{C}$,

$$f(z) = \exp(\bar{z}z) - 3\bar{z}z$$

Determine at which points $z_0 \in \mathbb{C}$ the function is holomorphic; that means, the complex derivative

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists.

Solution. The function f has continuous partial derivatives. Therefore, we only have to check at which points z_0 the Cauchy-Riemann equations hold, or equivalently, that

$$\frac{\partial f}{\partial \bar{z}}(z_0) = 0.$$

Clearly

$$\frac{\partial f}{\partial \bar{z}}(z_0) = z_0 \exp(\bar{z}_0 z_0) - 3z_0 = z_0(\exp(|z_0|^2) - 3)$$

so for the expression to be 0, either $z_0 = 0$ or $\exp |z_0|^2 = 3 \Leftrightarrow |z_0| = \log \frac{3}{2}$.

3. Find all entire functions $f : \mathbb{C} \to \mathbb{C}$ such that $f'(0) = 0$ and

$$|f(z)| \leq |z|^2 + 3 \qquad \text{for all } z \in \mathbb{C}.$$

Solution. Let $a_0 = f(0)$. Clearly $|a_0| \leq 3$. Consider the function

$$g(z) = \frac{f(z) - a_0}{z^2}, \quad z \in \mathbb{C}\backslash\{0\}.$$

Since $f(z) - a_0$ has a zero of order 2 in 0, g has a removable singularity in zero, and therefore extends to an entire function on $\mathbb{C}$. Moreover, g is bounded on $\mathbb{C}$, since

$$|g(z)| \leq 1 + \frac{6}{|z|^2}$$

for $|z| \geq 1$. Then g is entire and bounded, hence identical to a constant a_2 by Liouville's Theorem. Since $\lim_{z\to\infty} |g(z)| \leq 1$, $|a_2| \leq 1$. Altogether, f is a polynomial of degree 2, $f(z) = a_0 + a_1 z + a_2 z^2$, with $|a_0| \leq 3$, $a_1 = 0$ and $|a_2| \leq 1$. Conversely, each polynomial of this form clearly satisfies the inequality.

4. Consider the polynomial

$$p(z) = 3z^4 + z^3 + z^2 + z + 4.$$

Determine the number of zeros of p (counted with multiplicity) in the upper right quadrant Q of $\mathbb{C}$, that is,

$$Q = \{z \in \mathbb{C} : \operatorname{Re} z > 0, \operatorname{Im} z > 0\}.$$

Solution. By Rouché's Theorem, two analytic functions f and $f+g$ have the same number of zeroes inside the piecewise smooth closed simple curve γ, provided that $|f(z)| > |g(z)|$ for all $z \in \gamma$.
For $R > 2$, let γ_R be the closed positively oriented quartercircle of radius R around 0 through Q, that is,

$$\gamma_R = [0, R] \cup \{Re^{it} : 0 \leq t \leq \frac{\pi}{2}\} \cup [iR, 0]$$

Let $f(z) = 3z^4 + 4$ and $g(z) = z + z^2 + z^3$. Then

$$|f(z)| \geq 3R^4 - 4 > R^3 + R^2 + R \geq |g(z)|$$

for all z with $|z| = R$, and

$$|f(z)| = 3|z|^4 + 4 > |z|^3 + |z|^2 + |z| \geq |g(z)|$$

for all z on the real or imaginary axis. In particular,

$$|f(z)| > |g(z)| \text{ for all } z \in \gamma_R$$

therefore f and $p = f + g$ have the same number of zeros inside γ_R, namely 1. This is true for all $R > 2$. Thus p has one zero in the upper right quadrant.

5. Consider the function f given by

$$f(z) = \frac{2z+1}{z^2(z+1)^2} \qquad (z \in \mathbb{C}\backslash\{0, -1\}).$$

Determine the Laurent series expansion of the function f around 1 which converges in $z = -\frac{1}{2}$. For both the principal and the holomorphic part of the Laurent series expansion, determine the domain of convergence.
Solution. For $z \in \mathbb{C}\backslash\{0, -1\}$,

$$f(z) = \frac{2z+1}{z^2(z+1)^2} = \frac{1}{z^2} - \frac{1}{(z+1)^2} = \frac{1}{((z-1)+1)^2} - \frac{1}{((z-1)+2)^2}$$

$$\begin{aligned}\frac{1}{((z-1)+1)^2} &= \frac{1}{(z-1)^2}\frac{1}{(\frac{1}{z-1}+1)^2} = \frac{1}{(z-1)^2}\sum_{n=0}^{\infty}(-1)^n(n+1)\frac{1}{(z-1)^n} \\ &= \sum_{n=2}^{\infty}(-1)^n(n-1)\frac{1}{(z-1)^n} \qquad (|z-1| > 1)\end{aligned}$$

$$\frac{1}{((z-1)+2)^2} = \frac{1}{4}\frac{1}{(\frac{z-1}{2}+1)^2} = \sum_{n=0}^{\infty}(-1)^n\frac{1}{2^{n+2}}(n+1)(z-1)^n \qquad (|z-1| < 2),$$

Hence

$$f(z) = \sum_{n=-\infty}^{-2}(-1)^{n+1}(n+1)(z-1)^n + \sum_{n=0}^{\infty}(-1)^{n+1}\frac{1}{2^{n+2}}(n+1)(z-1)^n$$

converges on the annulus $A_{1,2}(1)$, the principal part $\sum_{n=-\infty}^{-2}(-1)^{n+1}(n+1)(z-1)^n$ converges on $A_1(1) = \{z \in \mathbb{C} : |z-1| > 1\}$ and the holomorphic part $\sum_{n=0}^{\infty}(-1)^n\frac{1}{2^{n+2}}(n+1)(z-1)^n$ converges on $D_2(1) = \{z \in \mathbb{C} : |z-1| < 2\}$.

6. Compute the integral

$$\int_{-\infty}^{\infty} \frac{x+2}{x^3+x} dx.$$

Solution. We use the Residue theorem.

Observe that $f(z) = \frac{z+2}{z(z^2+1)}$ has a simple pole in the upper half plane, namely at $z_0 = i$, and a simple pole at $z = 0$. Let $\gamma_{R,\varepsilon}$ denote the closed path consisting of the interval $[-R, -\varepsilon]$, the semicircle S_ε in the upper half-plane with radius ε around 0 in negative direction, the interval $[\varepsilon, R]$ and the semicircle S_R in the upper half-plane with radius R in positive direction. Then by the Residue Theorem,

$$\begin{aligned}
&\int_{\gamma_{R,\varepsilon}} f(z)dz = 2\pi i \operatorname{res}_{z_0} f \\
=\ & 2\pi i \lim_{z\to i} (z-i)f(z) \\
=\ & 2\pi i \lim_{z\to i} \frac{z+2}{z(z+i)} \\
=\ & 2\pi i \frac{i+2}{i(2i)} = \frac{\pi(i+2)}{i} = (1-2i)\pi.
\end{aligned}$$

Furthermore,

$$\lim_{R\to\infty} \left| \int_{S_R} f(z)dz \right| \leq \lim_{R\to\infty} \pi R \frac{R+2}{R(R^2-1)} = 0$$

and

$$\lim_{\varepsilon\to 0} \int_{S_\varepsilon} f(z)dz = \pi i \operatorname{res}_0 f = \pi i 2.$$

Hence

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{x+2}{(x^3+x^2)} dx &= \lim_{R\to\infty,\varepsilon\to 0} \int_{-R}^{-\varepsilon} \frac{x+2}{(x^3+x^2)} dx + \int_{\varepsilon}^{R} \frac{x+2}{(x^3+x^2)} dx \\
&= \lim_{R\to\infty,\varepsilon\to 0} \int_{\gamma_{\varepsilon,R}} f(z)dz - \int_{S_R} f(z)dz + \int_{S_\varepsilon} f(z)dz \\
&= (1-2i)\pi + \pi i 2 = \pi.
\end{aligned}$$