LUNDS UNIVERSITET

Tentamensskrivning
Analytiska funktioner
Tisdag den 10 januari 2017
Skrivtid: 14.00–19.00

Matematikcentrum
Matematik NF

Inga hjälpmedel. Använd institutionens papper, skriv på bara den ena sidan och högst en uppgift på varje papper. Skriv tydligt, ge klara och kortfattade motiveringar, rita gärna figur i förekommande fall och ge tydliga svar. Fyll i omslaget fullständigt och skriv initialer på varje papper.

1. Consider the function κ defined by the convergent power series

$$\kappa(z) = \sum_{k=1}^{\infty} k z^k, \quad z \in \mathbb{D},$$

in the open unit disc

$$\mathbb{D} = \{z \in \mathbb{C} : \ |z| < 1\}$$

in the complex plane $\mathbb{C}$. Show that κ continues analytically to an analytic function in the punctured plane $\Omega = \mathbb{C} \setminus \{1\}$. Calculate also the Laurent series expansion for κ in Ω.

2. Consider the polynomial

$$p(z) = 2z^2\bar{z}^3 - 9z\bar{z}^2 + z^2 + 12\bar{z} + 1$$

in z and $\bar{z}$. Determine at what points $z_0 \in \mathbb{C}$ the complex derivative $p'(z_0)$ exists in the usual sense that

$$p'(z_0) = \lim_{z \to z_0} \frac{p(z) - p(z_0)}{z - z_0}$$

as a limit in the complex plane.

3. Let f be analytic in a punctured neighborhood of a point $a \in \mathbb{C}$ and assume that f has a simple pole at a. Prove that

$$\mathrm{Res}(f; a) = \lim_{z \to a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \Big((z-a)^m f(z) \Big)$$

for $m \geq 1$, where $\mathrm{Res}(f; a)$ is the residue for f at a.

4. Calculate the integral

$$\int_0^{\pi} \frac{\cos(\theta)}{2 + \cos(\theta)} \, d\theta.$$

Var god vänd!

5. Denote by Aut($\mathbb{D}$) the set of all analytic functions φ in the open unit disc $\mathbb{D}$ mapping $\mathbb{D}$ one-to-one onto itself. Assume that $\varphi \in \mathrm{Aut}(\mathbb{D})$ has two (2) distinct fixed points in $\mathbb{D}$. Prove that

$$\varphi(z) = z \quad \text{for all} \quad z \in \mathbb{D}.$$

Here by a fixed point for φ in $\mathbb{D}$ is meant a point $z \in \mathbb{D}$ such that $\varphi(z) = z$.

6. Let $\mathbb{T}$ be the unit circle in the complex plane. Find an entire harmonic function u such that

$$u(e^{i\theta}) = \cos^2(\theta)$$

for $e^{i\theta} \in \mathbb{T}$, or prove that no such function u exists.