LUNDS UNIVERSITET

Matematikcentrum
Matematik NF

Tentamensskrivning
Analytiska funktioner
Lördag den 17 december 2016
Skrivtid: 8.00–13.00

Inga hjälpmedel. Använd institutionens papper, skriv på bara den ena sidan och högst en uppgift på varje papper. Skriv tydligt, ge klara och kortfattade motiveringar, rita gärna figur i förekommande fall och ge tydliga svar. Fyll i omslaget fullständigt och skriv initialer på varje papper.

1. Consider the function g defined by the convergent power series

$$g(z) = \sum_{k=0}^{\infty} (k+1) z^k, \quad z \in \mathbb{D},$$

in the open unit disc

$$\mathbb{D} = \{z \in \mathbb{C} : \ |z| < 1\}$$

in the complex plane $\mathbb{C}$. Show that g continues analytically to an analytic function in the punctured plane $\mathbb{C} \setminus \{1\}$. Calculate also the Laurent series expansion for g in the exterior disc

$$\Omega = \{z \in \mathbb{C} : \ |z| > 1\}.$$

2. Find an analytic function f mapping the open unit disc $\mathbb{D}$ one-to-one onto the disc

$$D = \{z \in \mathbb{C} : \ |z - 1 + 3i| < 2\}$$

such that $f(0) = 2 - 2i$ and $f'(0) > 0$.

3. Let u and v be two complex-valued C^1-functions in an open set Ω in the complex plane such that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

there. Determine whether it follows from these assumptions that the function

$$f = u + iv$$

is analytic in Ω. Give a proof or counter example.

4. Let f be analytic in a punctured neighborhood of a point $a \in \mathbb{C}$ and assume that f has a simple pole at a. Prove that

$$\operatorname{Res}(f; a) = \lim_{z \to a} \frac{d}{dz}\Big((z-a)^2 f(z)\Big),$$

where $\operatorname{Res}(f; a)$ is the residue for f at a.

Var god vänd!

5. Calculate the integral

$$\int_0^\infty \frac{\log x}{(x+1)^2}\, dx,$$

where $\log : (0,\infty) \to \mathbb{R}$ is the usual logarithm function defined by $d\log(x)/dx = 1/x$ for $x > 0$ and $\log(1) = 0$.

6. Let f be an entire analytic function such that

$$|f(z)| \to \infty \quad \text{as } |z| \to \infty.$$

Prove that f is a polynomial.