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OPTIMERING
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1. a) Exercise 4.8, p. 128.

b) Lemma 1, p. 43.

c) The implication is true. The proof is similar to Exercises 4.14–4.17. The system

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 1 & 2 & -1 \end{bmatrix} \lambda = \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix}$$

has the solution $\lambda = [2, 0, 1]^T \geq 0$. Use Farkas.

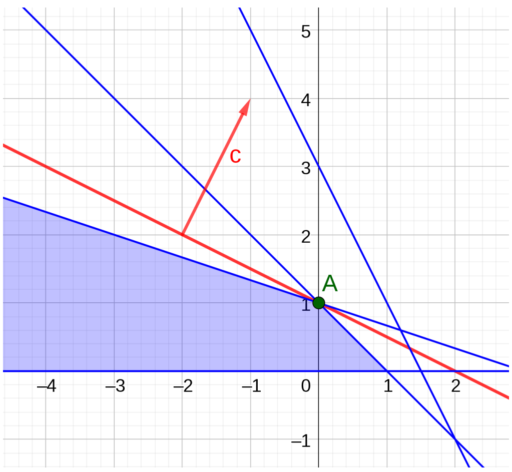
2. a) The minimum is at the stationary point $(1, -1)$ as the function is convex.

b) One possible auxiliary function is $q_\mu(x, y) = f(x, y) + \mu \max\{0, -y\}^2$. Since the optimal point in 2a) is infeasible and is taken as a starting point, all the iterations are going to be infeasible as well (see the discussion on the page 317), i.e. $\max\{0, -y\}^2 = y^2$ and, hence, $q_\mu(x, y) = f(x, y) + \mu y^2$. The stationary point is $\left(\frac{2\mu+1}{4\mu-1}, -\frac{3}{4\mu-1}\right) \to \left(\frac{1}{2}, 0\right)$ as $\mu \to +\infty$. Convexity of q_μ and Theorem 1, p. 316 approve optimality of the limit.

3. a) The dual problem is

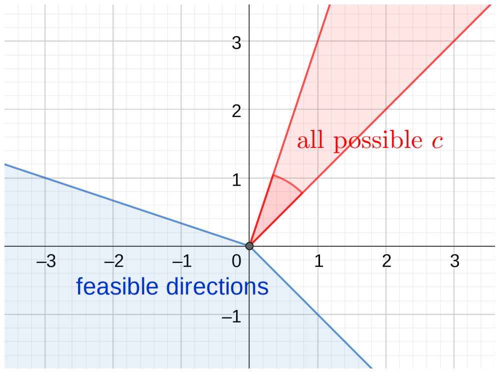
$$\max\,(c_1 y_1 + c_2 y_2) \qquad \text{subject to} \qquad \begin{cases} y_1 + 3y_2 & \leq & 3, \\ y_1 + y_2 & \leq & 1, \\ 2y_1 + y_2 & \leq & 3, \\ y_2 & \geq & 0. \end{cases}$$

Draw the dual feasible set (blue) and a level set of the dual function with $c = (1, 2)$ (red). The optimal point is $y = (0, 1)$.



The CSP for this dual solution gives $x_3 = 0$, $x_1 + x_2 = 1$ and $3x_1 + x_2 = 2$, with the primal solution being $x = (1/2, 1/2, 0)$.

b) The CSP once again for the given primal solution from 3a) ensures that the dual solution $y = (0, 1)$ is unique. The set of all vectors c that the dual problem has maximum at $(0, 1)$ is the dual cone (see p. 144 for definition) to the cone of all feasible directions at A. In other words, the cone of all possible c is such that the maximum of $c^T y$ is attained at A.



4. a) Convex as a composition of linear $h(x, y) = x + y$ and convex $f(t) = \max\{t^2, t^3\}$.

b) Convex. $2^x \cdot 4^{y^2} = 2^{x+2y^2}$. Composition of convex and increasing 2^t and convex $x + 2y^2$.

c) The quadratic form $x^T H x = (x_1 + x_2 - x_3)^2 + (a - 1)x_3^2$ is convex if and only if $a \geq 1$ (i.e. if and only if H is positive semidefinite). Hence, for $a \geq 1$ the level subset $\{x^T H x \leq 1\}$ is convex too. On the other hand, if $a < 1$ then the set is not convex since the intersection of the set with the line $x_1 = 2$, $x_2 = x_3 = t$ is not convex. It is easy to see that the intersection satisfies $4 + (a - 1)t^2 \leq 1$, which is true for large $|t|$ (as $a - 1$ is negative) and not true for $t = 0$. A counterexample is then the points $(2, t, t)$ and $(2, -t, -t)$ that are in the set, but the midpoint is not. Combining the two arguments gives the answer: $a \geq 1$.

5. Prove that the minimum exists. The origin is a CQ point. Two KKT points $(\pm\sqrt{2}, 1, 1)$ that are the minimum points.

6. a) The dual function is

$$\Theta(u) = \begin{cases} -\dfrac{4}{u+1} - \dfrac{(3-u)^2}{4}, & \text{if } 0 \leq u \leq 1, \\ -\infty, & \text{otherwise.} \end{cases}$$

The derivative $\Theta'(u)$ is positive on $[0, 1]$, hence, the maximum of Θ is at $\bar{u} = 1$. For this $\bar{u}$ we have $\bar{y} = \bar{z} = 1$. To find $\bar{x}$ we have to use the CSP

$$\bar{u}(\bar{z} - \bar{x}^2 + \bar{y}^2) = 0$$

which gives $\bar{x} = \pm\sqrt{2}$. Check no duality gap to confirm the solution.

b) Exercise A.2b).