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OPTIMERING
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1. a) (i) E.g. Rosenbrock function or $f(x,y) = 10^{10}x^2 + 10^{-10}y^2$,
(ii) E.g. $f(x,y) = \sqrt{x^2+y^2}$.

b) Solving $\min \|Ax-b\|$ is the same as solving $\min \|Ax-b\|^2$. This is the least square problem. Solve the normal equation $A^TAx = A^Tb$.
Answer: $x = (A^TA)^{-1}A^Tb = \begin{bmatrix}1 & 1/3\end{bmatrix}^T$.

c) Start with checking if the minimum exists. On the boundary $y = 1-x$, set it into the function to get $f(x, 1-x) = x - x^2$. Tell Alice that the problem makes no sense as the minimum does not exist.

2. a) Calculate the Hessian and the principal minors

$$H = 2\begin{bmatrix}4 & 1 & 1\\ 1 & 1 & a\\ 1 & a & 1\end{bmatrix} = 2\tilde{H}, \quad \det \tilde{H}_1 = 4,\ \det \tilde{H}_2 = 3,\ \det \tilde{H} = -2(a-1)(2a+1)$$

and combine the sufficient Sylvester with the necessary Sylvester.
Answer: $a \in [-1/2, 1]$.

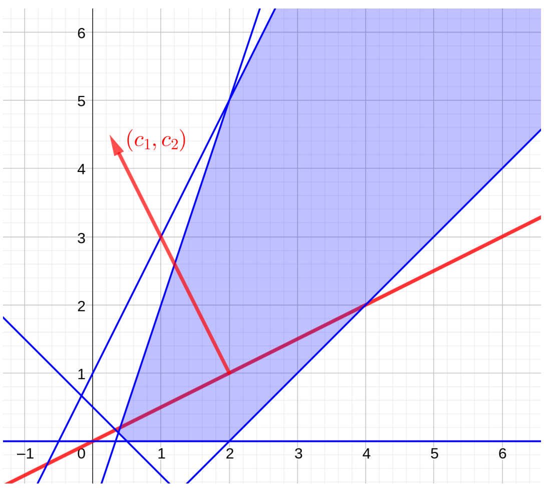
b) Solve $d_1^T H d_2 = 0$ and then $\begin{bmatrix}d_1^T\\ d_2^T\end{bmatrix} H d_3 = 0$ (many solutions).
Take e.g. $d_2 = \begin{bmatrix}0 & 1 & -1\end{bmatrix}^T$ and $d_3 = \begin{bmatrix}-1 & 2 & 2\end{bmatrix}^T$.

3. a) See [Theorem 6, p.182].

b) The dual problem is

$$\max\,(c_1y_1 + c_2y_2) \qquad \text{subject to} \qquad \begin{cases} y_1 - y_2 & \le & 2,\\ -2y_1 + y_2 & \le & 1,\\ 3y_1 - y_2 & \ge & 1,\\ 2y_1 + 2y_2 & \ge & 1,\\ y_1 & \ge & 0,\\ y_2 & \ge & 0.\end{cases}$$

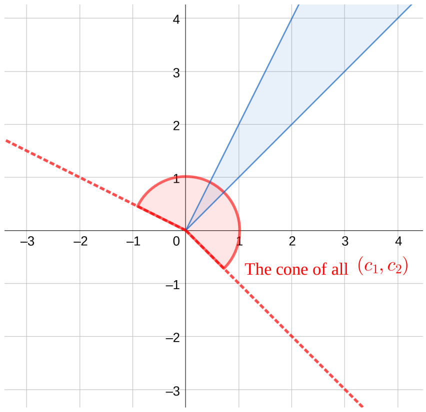
Draw the dual feasible set (it is not empty!) and a level set of the dual function.



From the weak duality result we have that

$$c_1 y_1 + c_2 y_2 \leq 2x_1 + x_2 + x_3 + x_4$$

for any feasible x and y. Hence, the primal set is empty if and only if the dual function is unbounded from above, i.e. the dual problem has an infinite "maximum". From the graphical argument it becomes clear that the vector (c_1, c_2) has to lay in the cone on the picture.



4. a) (i) False. Take e.g. $f(x) = x^2 - 1$, $g(x) = 0$.

(ii) True. $\phi' > 0$ implies (strictly) increasing, then Lemma 2(4), p. 211.

(iii) True, $\min\{f(x), g(x)\} = -\max\{-f(x), -g(x)\}$. The functions $-f$, $-g$ are convex, max of convex if convex, minus convex is concave.

b) $(\sqrt{x} + \sqrt{y})^2 \geq 2 \quad \Leftrightarrow \quad \sqrt{x} + \sqrt{y} \geq \sqrt{2} \quad \Leftrightarrow \quad (-\sqrt{x}) + (-\sqrt{y}) \leq -\sqrt{2}.$

The function $f(x) = -\sqrt{x}$ is convex for $x \geq 0$, sum of convex is convex, a sublevel set of a convex is convex.
Answer: the set is convex.

5. Minimize $x^2 + y^2 + z^2$ subject to $x^2 + y^2 - z^2 \geq 1$ and $x^2 + y^2 + z = 1$. Prove that the minimum exists, e.g. use $(1, 0, 0)$ as a feasible point and add $x^2 + y^2 + z^2 \leq 1$ to the constraints to make the new set compact. No CQ points. KKT points: $u = 1$, $v = 0$, $z = 0$, $x^2 + y^2 = 1$ (the whole circle of KKT points). The minimum is 1.

6. a) The dual function is

$$\Theta(u) = \begin{cases} 2u, & \text{if } 0 \leq u \leq 3, \\ -\infty, & \text{otherwise.} \end{cases}$$

The maximum of Θ is at $\bar{u} = 3$. For this $\bar{u}$ we have

$$L(x, y, 3) = x^2 - 2xy + y^2 + 6 = (x - y)^2 + 6$$

which attains minimum for $\bar{x} = \bar{y}$. For a saddle point the CSP has to hold, which gives $\bar{x}\bar{y} = 2$ (as $\bar{u} = 3 > 0$), hence, $\bar{x} = \bar{y} = \sqrt{2}$.
Answer: the saddle point is $(\bar{x}, \bar{y}, \bar{u}) = (\sqrt{2}, \sqrt{2}, 3)$.

b) The definition of a convex function tells us that for any two points the graph is below the secant within the interval, and above the secant outside the interval. Assume that the function is not a constant, hence it is possible to find two points such that the secant has a non-zero slope. For example, assume that the slope is positive. Then to the right of B the given convex function has to grow at least not slower than the secant. The secant grows to the infinity, therefore, the function must go to the infinity as well. It contradicts the assumption that the function is bounded from above. Similar argument if the slope is negative, consider what's going on at minus infinity.