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OPTIMERING
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1. a) The Hessian H must be positive semidefinite. Necessarily,

$$\det H = -(a+2)^2 \geq 0 \quad \Leftrightarrow \quad a = -2.$$

For this value of a we can complete the squares

$$x^T Hx = (x_1 - 2x_2 + x_3)^2 \geq 0, \ \forall x \in \mathbb{R}^3,$$

hence, the matrix is positive semidefinite by definition.
Answer: $a = -2$.

b) If the minimum exists then it is a stationary point of f. The function is convex, therefore, a stationary point is the global minimum. Thus, the minimum exists $\Leftrightarrow$ a stationary point exists $\Leftrightarrow$ there exists a solution to

$$\nabla f(x,y,z) = 2(x+y+z)\begin{bmatrix}1\\1\\1\end{bmatrix} + \begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \quad \Leftrightarrow \quad a = b = c.$$

Answer: $a = b = c$.

c) See the book, Lemma 1, p. 43.

2. a) The function is convex iff $-\ln(y - x^2)$ is convex. Try $h(x,y) = x^2 - y$ convex, $g(t) = -\ln|t|$, $t < 0$, increasing ($g'(t) = -\frac{1}{t} > 0$) and convex ($g''(t) = \frac{1}{t^2} > 0$), hence, $g(h(x,y))$ is convex.
Answer: Yes.

b) Yes, $q_\epsilon \to +\infty$ when (x,y) approaches the boundary $y = x^2$.

c) The minimum of q_ϵ on D_q exists because the function is convex and there is a feasible stationary point

$$\nabla q_\epsilon = \begin{bmatrix}1 + \epsilon\frac{2x}{y-x^2}\\ 1 - \epsilon\frac{1}{y-x^2}\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \quad \Leftrightarrow \quad \begin{cases} y - x^2 + 2x\epsilon = 0, \\ y - x^2 - \epsilon = 0. \end{cases} \quad \Leftrightarrow \quad \begin{cases} x = -\frac{1}{2}, \\ y = \frac{1}{4} + \epsilon, \end{cases}$$

which is the minimum of q_ϵ. It converges to $(-1/2, 1/4)$, hence, it is the global minimum of the constrained problem.

3. a) The dual problem is

$$\min\,(6y_1 + 7y_2 + 2y_3) \qquad \text{subject to} \qquad \begin{cases} y_1 + 3y_2 + 3y_3 \geq 1, \\ 3y_1 + 5y_2 + y_3 \geq 4, \\ y_1 + y_2 + y_3 \geq 3, \\ y_2 \geq 0, \ y_3 \leq 0. \end{cases}$$

The CSP gives $y_2 = y_3 = 0$ and $y_1 + y_2 + y_3 = 3$, that is $y = (3,0,0)$. It is dual feasible, hence, both solutions are optimal.

b) The statement 2 is equivalent to

$$A^T y = 0, \;\; b^T y > 0$$

has no solution. Rewriting it as the standard (first) Farkas alternative

$$\begin{bmatrix} A^T \\ -A^T \end{bmatrix} y \le 0, \quad b^T y > 0$$

makes the second alternative

$$\begin{bmatrix} A & -A \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = b, \quad \begin{bmatrix} u \\ v \end{bmatrix} \ge 0$$

being solvable. Denote $x = u - v$ to conclude that it is equivalent to the statement 1.

4. a) Since all g_k are convex we have

$$g(\lambda z_1 + (1-\lambda) z_2) = \begin{bmatrix} g_1(\lambda z_1 + (1-\lambda) z_2) \\ g_2(\lambda z_1 + (1-\lambda) z_2) \\ \vdots \\ g_n(\lambda z_1 + (1-\lambda) z_2) \end{bmatrix} \le \begin{bmatrix} \lambda g_1(z_1) + (1-\lambda) g_1(z_2) \\ \lambda g_2(z_1) + (1-\lambda) g_2(z_2) \\ \vdots \\ \lambda g_n(z_1) + (1-\lambda) g_n(z_2) \end{bmatrix} = \lambda g(z_1) + (1-\lambda) g(z_2).$$

Now we have convexity of h by definition from

$$\underbrace{f(g(\lambda z_1 + (1-\lambda) z_2))}_{h(\lambda z_1 + (1-\lambda) z_2)} \overset{(1)}{\le} f(\lambda g(z_1) + (1-\lambda) g(z_2)) \overset{(2)}{\le} \lambda \underbrace{f(g(z_1))}_{h(z_1)} + (1-\lambda) \underbrace{f(g(z_2))}_{h(z_2)}.$$

where (1) follows from the above relation and f being coordinate-wise increasing, and (2) follows from convexity of f.

b) The function h_1 is convex since

$$h_1(x,y,z) = \sqrt{x^6 + y^6 + z^6} = \sqrt{(|x|^3)^2 + (|y|^3)^2 + (|z|^3)^2}$$

can be split as $h = f \circ g$ where

$$f(g_1, g_2, g_3) = \sqrt{g_1^2 + g_2^2 + g_3^2} = \|g\|, \qquad g(x,y,z) = (|x|^3, |y|^3, |z|^3).$$

Here all $g_k(x,y,z)$ are convex and f is convex and coordinate-wise increasing.

The function h_2 is not convex. Set $y = z = 0$ to get the restriction $\sqrt[3]{|x|}$ that is not convex.

c) The Hessian is

$$H = \begin{bmatrix} 2 & 4y \\ 4y & 4x + 12(1+a)y^2 \end{bmatrix}.$$

It is positive-semidefinite in the set iff

$$\det H = 8(x + (1+3a)y^2) \ge 0, \quad \forall x \ge 0.$$

It is equivalent to $1 + 3a \ge 0 \Leftrightarrow a \ge -\frac{1}{3}$.
Answer: $a \ge -\frac{1}{3}$.

5. Existence of the minimum.
Use $x + y = 4$ and $x, y > 0$ to conclude that x, y are bounded from above.
Take $x = y = 2,\ z = 1/4$. Then it is enough to consider $xy + 2\ln z \le 4 - 2\ln 4 \le 2$ $\Rightarrow \ln z \le 1$, hence z is bounded from above as well.
Now use $xyz \ge 1$ to prove that x, y, z are bounded away from 0 from below. It makes the set compact. The minimum exists by Weierstrass theorem.

We set $X = \{(x, y, z)\colon x > 0,\ y > 0,\ z > 0\}$,

$$g(x, y, z) = 1 - xyz, \quad h(x, y, z) = x + y - 4.$$

CQ points. No such.
KKT points.

$$\begin{cases} y - uyz + v &= 0, & (1) \\ x - uxz + v &= 0, & (2) \\ \frac{2}{z} - uxy &= 0, & (3) \\ u(1 - xyz) &= 0, & (4) \\ u &\ge 0, & (5) \\ \text{feasibility} & & (6) \end{cases}$$

$u = 0$ makes (3) impossible. For $u > 0$ we get $xyz = 1$ and $u = 2$ from (3). The first two equations yields

$$v = y(2z - 1) = x(2z - 1) \quad \Rightarrow \quad (x - y)(2z - 1) = 0.$$

If $x = y$ then $x + y = 4 \Rightarrow x = y = 2$ and $xyz = 1 \Rightarrow z = 1/4$, hence, $(2, 2, 1/4)$ is a KKT point with $f = 4 - 2\ln 4 = 4 - 4\ln 2$.

If $2z = 1 \Leftrightarrow z = 1/2$ then $xy = 2$ from (3) which together with $x + y = 4$ implies that x, y solves the quadratic equation

$$t^2 - 4t + 2 = 0 \quad \Leftrightarrow \quad t = 2 \pm \sqrt{2}.$$

Another KKT point is $(2 \pm \sqrt{2}, 2 \mp \sqrt{2}, 1/2)$, $f = 2 - 2\ln 2$.

Answer: $(2 \pm \sqrt{2}, 2 \mp \sqrt{2}, 1/2)$.

6. a) Write down the Lagrange function

$$\begin{aligned} L(x, u) &= x^2 - 12x + y^2 + 2y + u_1(x^2 + y - 4) + u_2(-x^2 + y^2 + 1) = \\ &= (1 + u_1 - u_2)x^2 - 12x + (1 + u_2)y^2 + (2 + u_1)y + u_2 - 4u_1. \end{aligned}$$

Minimization w.r.t. $y \ge 0$ is clearly at $y = 0$.
Minimization w.r.t. x depends on:
if $1 + u_1 - u_2 \le 0$ then $\inf_x L = -\infty$.
if $1 + u_1 - u_2 > 0$ then the function L is convex in x and

$$L'_x = 2(1 + u_1 - u_2)x - 12 = 0 \quad \Leftrightarrow \quad x = \frac{6}{1 + u_1 - u_2},$$

hence it is the minimum.
It gives

$$\Theta(u) = \begin{cases} -\dfrac{36}{1+u_1-u_2} - 4u_1 + u_2, & \text{if } 0 \le u_2 < 1+u_1, \\ -\infty & \text{otherwise.} \end{cases}$$

Let us see if there is a stationary point w.r.t. u_1

$$\Theta'_{u_1} = \frac{36}{(1+u_1-u_2)^2} - 4 = 0 \quad \Leftrightarrow \quad 1+u_1-u_2 = 3 \quad \Leftrightarrow \quad u_1 = u_2 + 2.$$

This u_1 maximizes Θ as the dual function is concave. It is left to maximize w.r.t. $u_2 \ge 0$ for the found u_1

$$\Theta = -12 - 4(u_2+2) + u_2 = -20 - 3u_2.$$

The function is decreasing, hence, the maximum is at $u_2 = 0$. Therefore, $\bar{u}_2 = 0$, $\bar{u}_1 = 2$ and $\Theta(\bar{u}) = -20$. The candidate x, y from above is $\bar{x} = 6/3 = 2$, $\bar{y} = 0$ gives $f(\bar{x}, \bar{y}) = -20 = \Theta(\bar{u})$, thus, the optimal point is $(2, 0)$.

b)

$$\Theta(u,v) = \inf_{x \in X} L(x,u,v) \le L(x,u,v) = f(x) + \underbrace{u^T g(x)}_{\le 0} + v^T \underbrace{h(x)}_{=0} \le f(x).$$