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OPTIMERING
2018-04-11 kl 14-19

Answers and brief comments only for re-exams. No complete solutions.

1. a) E.g. $d_2 = [1,1,1]^T$, $d_3 = [1,0,-1]^T$.

b) The function is convex, then a stationary point is the global minimum. Calculate the stationary point as in Lemma 3, p. 71.

2. a) $\mathbf{x}^T H\mathbf{x} = 2x_2(x_1 + x_3)$ can take both positive (e.g. for $x_1 = x_2 = x_3 = 1$) and negative (e.g. for $x_1 = x_3 = 1$, $x_2 = -1$) values, hence, indefinite.

b) Check by Sylvester, that $\epsilon = 1$ does not work, but $\epsilon = 2$ is ok. The Modified Newton method is then

$$\mathbf{x}_{k+1} = \mathbf{x}_k - (H + 2I)^{-1}Hx_k = (H + 2I)^{-1}(H + 2I - H)\mathbf{x}_k = 2(H + 2I)^{-1}\mathbf{x}_k$$

where

$$2(H + 2I)^{-1} = \frac{1}{2}\begin{bmatrix} 3 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 3 \end{bmatrix}.$$

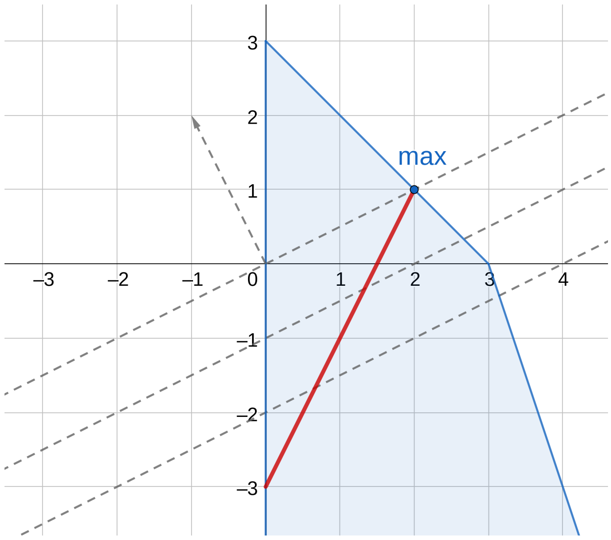
Thus, $\mathbf{x}_1 = [3,-4,3]^T$, $\mathbf{x}_2 = [10,-14,10]^T$, $\mathbf{x}_3 = [34,-48,34]^T$. Already after two steps it seems that the method diverges.

c) Taking $\mathbf{x}_0 = [1,-1,1]$ (that makes $\mathbf{x}^T H\mathbf{x}$ negative in 2a)) and $\mathbf{x} = t\mathbf{x}_0$, $t \to +\infty$, we get $f(\mathbf{x}) \to -\infty$, hence, the minimum does not exist.

3. a) The dual problem is

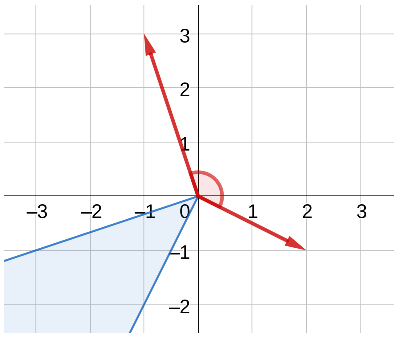
$$\max\,(-y_1 + 2y_2) \qquad \text{subject to} \qquad \begin{cases} y_1 + y_2 & \le & 3, \\ 2y_1 + y_2 & \le & 6, \\ 2y_1 - y_2 & = & 3, \\ y_1 \ge 0, & & \\ y_2 \text{ free.} & & \end{cases}$$

The optimal solution (see the picture below) is $(\bar{y}_1, \bar{y}_2) = (2,1)$. The CSP gives $x_2 = 0$ (the second dual constraint is not active) and the first primal inequality becomes equality. Solving the system, we get $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (1,0,-1)$.



The feasible set is the red segment.

b)



The set of all c is the red cone.

4. a)

- Not convex (try $y = z = 0$).
- Convex as $h(x, y, z) = x^2 + y^2 + z^2 \geq 0$ is convex and $g(t) = t^3$ is convex and increasing for $t \geq 0$.
- Convex as the Hessian is positive semidefinite.

b) Convex as an intersection of convex sets. The set $x_1^2 + x_2^2 + x_3^2 - x_4 \leq 0$ is convex since the function $x_1^2 + x_2^2 + x_3^2 - x_4$ is convex.

5. The minimum exists: the set is closed, the variables are bounded below as $-1 \leq \sin x \leq y \leq x$. To see they are bounded above as well, take any feasible point and combine $x + 2y \leq C$ with the inequalities above (e.g. use $-x \leq -y \leq 1$).

CQ point: $(0, 0)$.

KKT points: $(\frac{2\pi}{3} + 2\pi k, \frac{\sqrt{3}}{2})$ and $(\frac{4\pi}{3} + 2\pi k, -\frac{\sqrt{3}}{2})$, $k = 0, 1, 2, \ldots$.

The minimum is attained at the CQ point: $f(0, 0) = 0$.

(The least value at KKT points is $f(\frac{4\pi}{3}, -\frac{\sqrt{3}}{2}) = \frac{4\pi}{3} - \sqrt{3} > 4 - \sqrt{3} > 0$.)

6. a) The Lagrange function

$$L(x, y, u) = xy + ux^2 + (1 - u)y^2 + u$$

has the obvious minimum on X with respect to $x \geq 0$ at $x = 0$. The rest has minimum at $y = 0$ for $u \leq 1$, otherwise $-\infty$ as $y \to +\infty$. The dual function is $\Theta(u) = u$ for $u \leq 1$, otherwise $-\infty$. The solution to the dual problem is $\bar{u} = 1$, and $\Theta(1) = 1$. The minimum of L with respect to x for $\bar{u} = 1$ is at $\bar{x} = 0$. To obtain $\bar{y}$ we need the Complementary Slackness principle: $\bar{u}(1 + \bar{x}^2 - \bar{y}^2) = 0 \Rightarrow \bar{y}^2 = 1 + \bar{x}^2 = 1 \Rightarrow \bar{y} = 1$. Testing the duality gap for the candidate:

$$\Theta(1) = f(0, 1) = 1.$$

No gap, hence, the optimal solution is $(0, 1)$.

b) Assume $f(c) = f'(c) = 0$ and take any $x \in [a, b]$. Then for $\lambda \in (0, 1)$ we have by the definition of a convex function that

$$f(c + \lambda(x - c)) = f(\lambda x + (1 - \lambda)c) \leq \lambda f(x) + (1 - \lambda) \underbrace{f(c)}_{=0} = \lambda f(x).$$

Dividing by λ and taking the limit as $\lambda \to 0^+$

$$\frac{f(c + \lambda(x - c))}{\lambda} = \underbrace{\frac{f(c + \lambda(x - c)) - f(c)}{\lambda}}_{\to f'(c)(x-c)=0} \leq f(x).$$

The left hand side converges (by the definition of a derivative) to $f'(c)(x-c) = 0$, therefore, $f(x) \geq 0$.

(See also Lecture 7, page 6.)