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MATEMATIK

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OPTIMERING
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1. a) CQ points are shown as A and B below. Graphically: they are the only points that share the same tangent line for both active constraints. Furthermore, the gradients of two active constraints are directed opposite to each other. Hence, they are CQ points. It is possible to solve the problem analytically as well, however, the calculation would take more time.

b) The set of feasible directions are the cone between the tangent lines to two active constraints at $(\sqrt{3}, 0)$ (including the one line and excluding the other one). The tangent lines go through the point $(\sqrt{3}, 0)$ and the point $B : (0, 3)$ and the point $(0, -3)$ respectively. The cone of all possible gradients ∇f is between the outer normals (=gradients to the active constraint functions at the point). Those gradients are parallel to the lines through the point and the point $A : (0, 1)$ and $(0, -1)$ respectively.

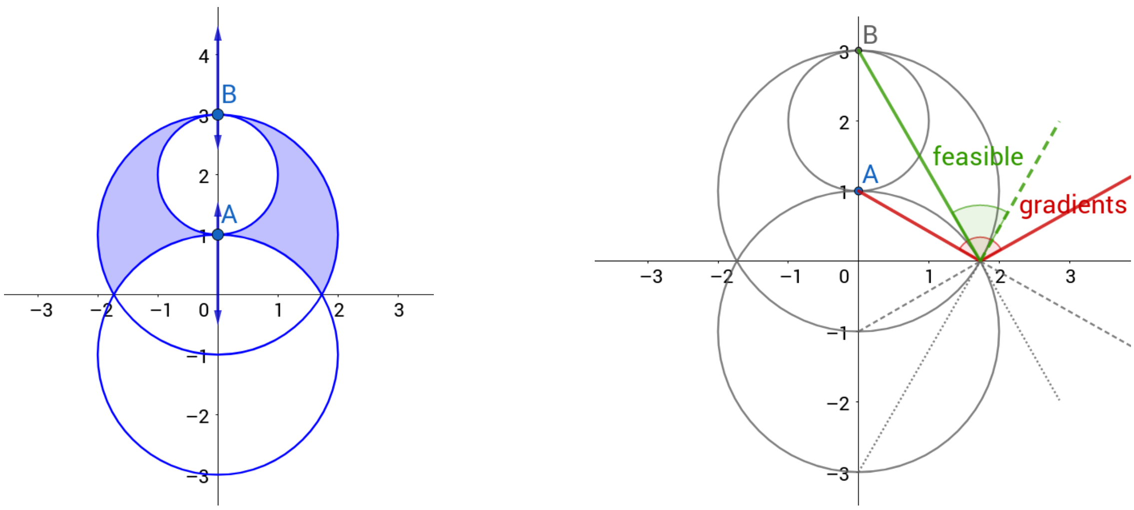

Figure 1: Drawing in 1a) to the left and in 1b) to the right

2. Consider the optimization problem

$$\min x^2 + 4y^2 \qquad \text{subject to } xy \geq 1.$$

a) $(1, 1)$ is feasible, add $x^2 + 4y^2 \leq 5$ to get a compact set and use the Weierstrass theorem. The problem is not convex (due to $g(x, y) = 1 - xy$). Simplified argument: the minimum is where the gradients to f and g are parallel (since g must be active at the minimum, otherwise we can go a bit towards the unconstrained min at $(0, 0)$ and make the functional value smaller):

$$\det \begin{bmatrix} x & y \\ 4y & x \end{bmatrix} = x^2 - 4y^2 = 0 \qquad \Longrightarrow \qquad x = \pm 2y.$$

Together with $xy = 1$ we have two solutions $\pm(\sqrt{2}, \frac{1}{\sqrt{2}})$. KKT method works too, of course, but is perhaps overkill here.

b) Add penalty e.g.

$$q_\mu(x,y) = x^2 + 4y^2 + \mu \max\{0, 1 - xy\}^2.$$

The penalty is zero at $(1,2)$. The function is *quadratic*, hence, the first Newton step gives the unconstraint minimum $(0,0)$ for any μ (no need to actually do any calculations here). The origin is a stationary point for q_μ for any μ

$$q_\mu(x,y) = x^2 + 4y^2 + \mu(xy-1)^2 \;\Rightarrow\; \nabla q_\mu = \begin{bmatrix} 2x + 2\mu(xy-1)y \\ 8y + 2\mu(xy-1)x \end{bmatrix} \;\Rightarrow\; \nabla q_\mu(0,0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

hence, we do not move anywhere on the second step for any μ.

c) Add barrier with $\epsilon = 1$, e.g.

$$q(x,y) = x^2 + 4y^2 - \ln(xy-1) \quad \left[\text{or} \quad qq(x,y) = x^2 + 4y^2 + \frac{1}{xy-1} \right].$$

Then $\nabla q(1,2) = (0, 15)^T$ and $d = -H^{-1}\nabla q(1,2) \approx (0.3, -1.8)^T$. It points much better to the minimum.

3. a) Check that the point is feasible for the primal problem. Construct the dual problem e.g.

$$\min\,(2y_1 + 6y_2 + 4y_3) \qquad \text{subject to} \qquad \begin{cases} 5y_1 + 9y_2 + 3y_3 & \geq \; 1, \\ 6y_1 + 3y_2 - y_3 & \geq \; 2, \\ y_1 + 2y_2 + y_3 & \geq \; 4, \\ y_1 \leq 0, \; y_3 \geq 0. \end{cases}$$

CSP gives that $y_1 = y_3 = 0$, hence $y_2 = 2$. Check that $y = (0,2,0)$ is dual feasible and conclude that both are optimal by CSP.

b) The second system can be written as the first Farkas alternative

$$\begin{cases} \begin{bmatrix} -A^T \\ A^T \\ -I \end{bmatrix} y & \leq \; 0, \\ -b^T y & > \; 0. \end{cases}$$

Then by Farkas theorem we have the second alternative as

$$\begin{cases} \begin{bmatrix} -A & A & -I \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} & = \; -b, \\ (u,v,w) & \geq \; 0. \end{cases}$$

Setting $x = u - v$ we get $-Ax - w = -b \iff Ax \leq b$ since $w \geq 0$.

4. a) See the book, Lemma 2, p. 121, and Exercise 4.8, p. 128.

b) The first function can be split as $g(h(\mathbf{x}))$ where $h(\mathbf{x}) = 1 + (x+y+z)^2$ convex ($x+y+z$ is affine and the squaring is convex) and $h(\mathbf{x}) \geq 1$, and $g(t) = t \ln t$ is convex and increasing for $t \geq 1$ (as $g'(t) = \ln t + 1 \geq 0$ and $g''(t) = \frac{1}{t} \geq 0$).
The second function is not convex: take the line $x = y = z$, then the restriction to the line is $8x^3$, which is not convex on $\mathbb{R}$ ($(8x^3)'' = 48x < 0$ for $x < 0$).

c) Divide $xy \geq x + y$ by xy to get (as $x > 0$, $y > 0$) the condition $\frac{1}{x} + \frac{1}{y} \leq 1$. It is the sublevel set of the convex function $\frac{1}{x} + \frac{1}{y}$, thus, convex.

5. Using convexity (the shortest solution): the objective function is convex (e.g. check the Hessian by Sylvester criterion)

$$\nabla\left(\frac{x^2+y^2}{x+y}\right) = \begin{bmatrix} \frac{x^2-y^2+2xy}{(x+y)^2} \\ \frac{-x^2+y^2+2xy}{(x+y)^2} \end{bmatrix}, \qquad \nabla^2\left(\frac{x^2+y^2}{x+y}\right) = \frac{4}{(x+y)^3}\begin{bmatrix} y^2 & -xy \\ -xy & x^2 \end{bmatrix},$$

the constraint $xy \geq 1$ is not convex, but if we divide it by $x > 0$, it becomes $\frac{1}{x} - y \leq 0$, which is convex. Hence, with $X = \{x > 0, y > 0\}$, the problem becomes convex. By sufficient condition, any KKT point is the global minimum. From the KKT conditions

$$\begin{cases} \frac{x^2-y^2+2xy}{(x+y)^2} - u\frac{1}{x^2} & = 0, \\ \frac{-x^2+y^2+2xy}{(x+y)^2} - u & = 0, \\ u\left(\frac{1}{x} - y\right) & = 0, \\ u \geq 0,\ x > 0,\ y > 0, \\ 1 - xy \leq 0, \end{cases} \quad \Leftrightarrow \quad \begin{cases} x^2 - y^2 + 2xy - u\frac{(x+y)^2}{x^2} & = 0, \\ -x^2 + y^2 + 2xy - u(x+y)^2 & = 0, \\ u(1 - xy) & = 0, \\ u \geq 0,\ x > 0,\ y > 0, \\ 1 - xy \leq 0, \end{cases}$$

we see that $u \neq 0$ (otherwise $4xy = 0$: not possible). Then $xy = 1$. It is quite easy to notice that $x = y = 1$ is a KKT point. If you do not think so, take the first equation minus the second one to get

$$2x^2 - 2y^2 + u(x+y)^2(1 - \frac{1}{x^2}) = 0 \quad \Leftrightarrow \quad 2x^4 - 2\underbrace{(xy)^2}_{=1} + u(x+y)^2(x^2-1) = 0.$$

Since $x^4 - 1 = (x^2-1)(x^2+1)$ we can factor out the common factor $x^2 - 1$ and obtain

$$(x^2-1)\Big(2(x^2+1) + u(x+y)^2\Big) = 0.$$

Since the second factor is stricktly positive, it makes $x^2 = 1$, thus, $x = 1$ (as $x > 0$) and $y = 1$. It is a KKT point, therefore, the global minimum by the sufficient KKT condition.

Alternative solution (without convexity): prove that min exists. The strict constraints are never active, that is, the set is closed (you may draw the set to see that the boundary is $xy = 1$, and it is, indeed, in the set.) Then we need boundedness only. Take e.g. $x = y = 2$ and add $x^2 + y^2 \leq 2(x+y)$ to the constraints. Complete the squares to see that the new set is bounded (the added constraint is a disc $(x-1)^2 + (y-1)^2 \leq 2$, i.e. bounded). Weierstrass gives the existence. With $X = \{x > 0, y > 0\}$ we have only one explicit constraint. No CQ points (verify!).

The single KKT point can be found similarly to the other solution: take the first KKT equation minus the second one

$$\begin{cases} \frac{x^2-y^2+2xy}{(x+y)^2} - uy &= 0, \\ \frac{-x^2+y^2+2xy}{(x+y)^2} - ux &= 0, \end{cases} \quad \Leftrightarrow \quad \begin{cases} x^2 - y^2 + 2xy - u(x+y)^2 y &= 0, \\ -x^2 + y^2 + 2xy - u(x+y)^2 x &= 0, \end{cases}$$

to get

$$2x^2 - 2y^2 + u(x-y)(x+y)^2 = 0.$$

Use $x^2 - y^2 = (x-y)(x+y)$ to factor out the common $x - y$

$$(x-y)(2(x+y) + u(x+y)^2) = 0.$$

The second factor has no zeros since $u \geq 0$ and $x, y > 0$, thus, no more KKT points except for $(x - y = 0,\ xy = 1 \implies)\ x = y = 1$.

6. a) The problem is not the easiest one. Set up the Lagrange function

$$L(x, y, z, u_1, u_2) = z^2 - zu_1(x+y) + u_1(x^2+y^2) + u_2(1-xy)$$

and minimise on z to get $z = \frac{u_1}{2}(x+y)$. We are left with a quadratic function to minimize

$$\begin{aligned} \inf_z L &= \left(u_1 - \frac{1}{4}u_1^2\right)x^2 + \left(-u_2 - \frac{u_1^2}{2}\right)xy + \left(u_1 - \frac{1}{4}u_1^2\right)y^2 + u_2 = \\ &= \begin{bmatrix} x \\ y \end{bmatrix}^T Q \begin{bmatrix} x \\ y \end{bmatrix} + u_2 \end{aligned}$$

where Q is the matrix of the quadratic form

$$Q = \begin{bmatrix} a & -b \\ -b & a \end{bmatrix} = \frac{1}{4}\begin{bmatrix} u_1(4-u_1) & -2u_2 - u_1^2 \\ -2u_2 - u_1^2 & u_1(4-u_1) \end{bmatrix}.$$

Note that if $a < 0$ (i.e. $u_1 > 4$) then $\inf L = -\infty$ (take $y = 0$, $x \to +\infty$).

Let us study the case $a \geq 0$. We are to minimize

$$q(x,y) = \begin{bmatrix} x \\ y \end{bmatrix}^T Q \begin{bmatrix} x \\ y \end{bmatrix} = ax^2 - 2bxy + ay^2$$

over $x > 0$, $y > 0$ given $a \geq 0$, $b \geq 0$.

Proposition 1 *The following statements are equivalent:*

1. *Q is positive semidefinite,*
2. *$q(1,1) \geq 0$,*
3. *$b \leq a$.*

Proof:
$1 \Rightarrow 2$: trivially by definition of positive definiteness.
$2 \Rightarrow 3$: trivially by

$$q(1,1) = a \cdot 1 - 2b \cdot 1 + a \cdot 1 = 2(a-b) \geq 0.$$

$3 \Rightarrow 1$: if $a = 0$ then $b = 0$, hence, Q is the zero matrix, and it is positive semidefinite.
If $a > 0$ then $a + b > 0$ and

$$\det Q = a^2 - b^2 = (a-b)(a+b) \geq 0.$$

Therefore, by Sylvester criterion Q is positive semidefinite. Q.E.D.

The Proposition gives us directly the following result

$$\inf_{x>0,\, y>0} q(x,y) = \begin{cases} 0 & \text{if } Q \text{ is pos.semidef.,} \\ -\infty & \text{otherwise.} \end{cases}$$

The otherwise-part: if Q is not positive semidefinite then $q(1,1) < 0$, and taking $x = y \to +\infty$ makes $q \to -\infty$. Moreover, Q is positive semidefinite if and only if

$$a - b \geq 0 \quad \Leftrightarrow \quad u_1(4-u_1) - 2u_2 - u_1^2 \geq 0 \quad \Leftrightarrow \quad u_2 \leq u_1(2-u_1).$$

Back to the Lagrange function minimization. We have

$$\Theta(u_1,u_2) = \inf L = \begin{cases} u_2 & \text{if } 0 \leq u_2 \leq u_1(2-u_1), \\ -\infty & \text{otherwise.} \end{cases}$$

Maximum is when (clearly) $u_2 = u_1(2-u_1)$. Maximizing $u_1(2-u_1)$ subject to $0 \leq u_1 \leq 2$ gives the optimal $u_1 = u_2 = 1$.

Unfortunately it does not give us unique x, y values for $\inf L$, so we have to plug the optimal $u_1 = u_2 = 1$ and $z = \frac{x+y}{2}$ into the Lagrange function to get all the candidates

$$L = \frac{3}{4}(x-y)^2 + 1.$$

Then the minimum is when $x = y$. Together with the Complementary Slackness $u_1(xy-1) = 0$, i.e. $xy = 1$, we finally obtain the candidate for the saddle point $x = y = 1$. Testing shows no duality gap. A pour man's solution would be to take the candidates $x = y = 1$ from the solution to Problem 5 and check the duality gap (as long as we have no gap, it is the optimal solution, and nobody cares how you manage to "guess" those values).

b) See the book.