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LÖSNINGAR
OPTIMERING
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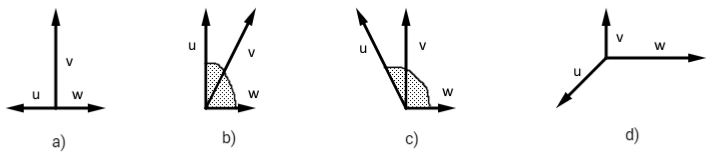
1. a)

- No. SD has orthogonal search directions, which is obviously not the case here.
- Yes. CD starts from minus the gradient, that is, orthogonal to the level curve, and finds the minimum after 2 iterations.
- No. Newton's method would find the minimum in one step.
- Yes. Quasi-Newton and CD have identical iterations for quadratic functions.

b) Answer: cases a) and d) violate the CQ condition (i.e. the point is a CQ point).



Case a): Yes, it is a CQ point. Two vectors u and w are already antiparallel ($u + w = 0$), thus all three are positively linearly dependent.

Case b) and c): No, all positive linear combinations span the pointed convex cone where the origin cannot be a non-trivial positive combination of the vectors.

Case d): Yes, it is a CQ point too because $-u$ belongs to the cone spanned by the other two vectors, so they are positively linearly dependent.

2. a) Let's calculate the Hessian

$$\nabla f = 3 \begin{bmatrix} x^2 - y \\ y^2 - x \end{bmatrix}, \qquad H = 3 \begin{bmatrix} 2x & -1 \\ -1 & 2y \end{bmatrix}$$

and find out if H is positive semidefinite in S. Note first that x cannot be equal to zero in S because $x = 0$ clearly violates $xy \geq 1$. Then we can apply Sylvester criterion to H in S:

$$\det H_1 = 2x > 0, \qquad \det H_2 = \det H = 4xy - 1 \geq 4 \cdot 1 - 1 = 3 > 0.$$

Hence, H is positive definite, and, therefore, the function f is convex.

b) Solving $\nabla f = 0$ gives two stationary points $(0, 0)$ and $(1, 1)$. The latter belongs to S, therefore, it is the global minimum in S since the function is convex.

Alternative solution (longer): The set S is closed, but not bounded. Let's take a point in S, for example, $(1, 1)$ (the easiest). The minimum if it exists would satisfy $f(x, y) = x^3 - 3xy + y^3 \leq 1 - 3 + 1 = -1$, so we can add this extra inequality to the constraints and build a new (smaller) set. The third order terms x^3 and y^3 at infinity grow faster then the second order xy, so for large $x \geq 0$ or $y \geq 0$ we get $f(x, y) \geq 0$, which means that those x, y are not going to satify the extra condition $f(x, y) \leq -1$, hence the new set is bounded. Apply now the Weierstrass theorem to conclude the existence of the minimum.

c) The function with penalty added is

$$q_\mu = x^3 - 3xy + y^3 + \mu \max\{1 - xy, 0\}^2 + \mu \max\{-x, 0\}^2.$$

Near the point $(1, 0)$ the last term is zero, and the function becomes

$$q_\mu = x^3 - 3xy + y^3 + \mu(1 - xy)^2 = x^3 - 3xy + y^3 + \mu(xy - 1)^2.$$

Calculate for $\mu = 3$ at $(x, y) = (1, 0)$:

$$\begin{aligned} \nabla q_3 &= 3\begin{bmatrix} x^2 - y \\ y^2 - x \end{bmatrix} + 2\mu(xy - 1)\begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} 3 \\ -9 \end{bmatrix} = 3\begin{bmatrix} 1 \\ -3 \end{bmatrix}, \\ \nabla^2 q_3 &= 3\begin{bmatrix} 2x & -1 \\ -1 & 2y \end{bmatrix} + 2\mu\begin{bmatrix} y^2 & 2xy - 1 \\ 2xy - 1 & x^2 \end{bmatrix} = 3\begin{bmatrix} 2 & -3 \\ -3 & 2 \end{bmatrix}. \end{aligned}$$

It makes the Newton direction to be

$$d = -(\nabla^2 q_3)^{-1}\nabla q_3 = -\begin{bmatrix} 2 & -3 \\ -3 & 2 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -7 \\ -3 \end{bmatrix}.$$

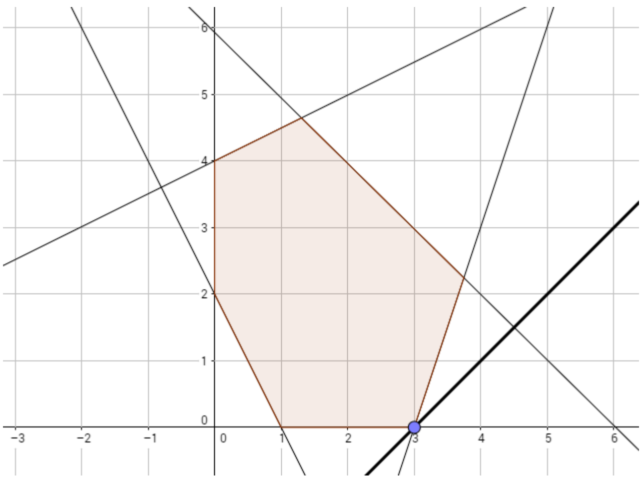
3. a) Change the second inequality sign

$$\min\,(9x_1 + 6x_2 + 8x_3 - 2x_4) \qquad \text{subject to} \qquad \begin{cases} 3x_1 + x_2 - x_3 - 2x_4 \geq 1, \\ -x_1 + x_2 + 2x_3 - x_4 \geq -1, \\ \text{all } x_k \geq 0 \end{cases}$$

and build the standard dual problem as

$$\max\,(y_1 - y_2) \qquad \text{subject to} \qquad \begin{cases} 3y_1 - y_2 \leq 9, \\ y_1 + y_2 \leq 6, \\ -y_1 + 2y_2 \leq 8, \\ -2y_1 - y_2 \leq -2, \\ \text{all } y_k \geq 0. \end{cases}$$

Graphical maximization results in the optimal solution $\bar{y} = (3, 0)$ with the optimal value being $\max = 3$.



Now we use the CSP to find the primal solution. For the dual optimal we see that it is only the first inequality among first four that becomes active. It means that the second, third and fourth inequalities are strict. The CSP gives then that $\bar{x}_2 = \bar{x}_3 = \bar{x}_4 = 0$. Furthermore, $\bar{y}_1 \neq 0$ implies equality in the first primal condition, that is

$$1 = 3\bar{x}_1 + \bar{x}_2 - \bar{x}_3 - 2\bar{x}_4 = 3\bar{x}_1 + 0 + 0 + 0 \qquad \Rightarrow \qquad \bar{x}_1 = \frac{1}{3}.$$

The vector $\bar{x}$ is primal feasible, hence, it is the optimal solution. Answer: the primal optimal $\bar{x} = \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 \end{pmatrix}$.

b) Let's calculate the dual problems to $\mathbf{P}_1$ and $\mathbf{P}_2$ as $\mathbf{D}_1$, $\mathbf{D}_2$ respectively

$$\mathbf{D}_1\colon \max b^T y \text{ subject to } \begin{cases} A^T y \leq c \\ y \geq 0 \end{cases}, \quad \mathbf{D}_2\colon \max [b^T\ 0] z \text{ subject to } \begin{cases} [A^T\ \ I] z = c \\ z \geq 0 \end{cases}.$$

To see that they are actually the same problem, we can partition z as $z = \begin{bmatrix} y \\ w \end{bmatrix}$ in the same way as the vector $\begin{bmatrix} b \\ 0 \end{bmatrix}$. Then the problem $\mathbf{D}_2$ may be rewritten as

$$\max b^T y \text{ subject to } \begin{cases} A^T y + w = c \\ y \geq 0,\ w \geq 0 \end{cases} \quad \Leftrightarrow \quad \begin{cases} w = c - A^T y \geq 0 \\ y \geq 0 \end{cases}.$$

After elimination the (slack) variable w the problem $\mathbf{D}_2$ becomes $\mathbf{D}_1$.

4. a) For the affine function $h(x) = Ax + b$ we have

$$\lambda h(x_1) + (1-\lambda) h(x_2) = \lambda(Ax_1 + b) + (1-\lambda)(Ax_2 + b) = A(\lambda x_1 + (1-\lambda x_2) + b = h(\lambda x_1 + (1-\lambda x_2)$$

for any vectors x_1, x_2 and scalar λ. It makes h convex by definition.

b) Consider the function $g(x, y) = \frac{x^2}{y}$ on the set $y > 0$. Calculate the Hessian

$$\nabla g = \begin{bmatrix} \frac{2x}{y} \\ -\frac{x^2}{y^2} \end{bmatrix} \quad \Rightarrow \quad \nabla^2 g = H = \begin{bmatrix} \frac{2}{y} & -\frac{2x}{y^2} \\ -\frac{2x}{y^2} & \frac{2x^2}{y^3} \end{bmatrix}.$$

Try Sylvester: $\det H_1 = \frac{2}{y} > 0$ for $y > 0$ and $\det H_2 = \frac{4x^2}{y^4} - \frac{4x^2}{y^4} = 0 \geq 0$. The sufficient (aka modified) Sylvester criterion ensures that H is positive semidefinite, i.e. the function g is convex. It makes easily even

$$g(y, z) = \frac{y^2}{z}, \qquad g(z, x) = \frac{z^2}{x}$$

be convex on $z > 0$, resp. $x > 0$. Therefore, the function

$$f(x, y, z) = g(x, y) + g(y, z) + g(z, x)$$

is convex on $X = \{x > 0, y > 0, z > 0\}$ as a sum of convex functions. Finally,

$$\Omega = \{(x, y, z) \in X \colon f(x, y, z) \leq 1\}$$

is convex as a sublevel set of a convex function.

c) The function g is convex and the function $s(x) = 0$ (the constant zero) is convex, then $h(x) = \max\{g(x), 0\}$ is convex (Lemma 2, p.211). Moreover, the function $f(t) = t^2$ is convex ($f''(t) = 2 > 0$) and increasing for $t \geq 0$ ($f'(t) = 2t \geq 0$) as well as $h(x) \geq 0$. Hence, by the same Lemma 2, the composition $f(h(x))$ of a convex increasing f and a convex h is convex.

5. The problem is to minimize

$$\min \sqrt{x^2 + y^2 + z^2} \quad \text{subject to } \frac{1}{xy} + \frac{1}{yz} \leq 1, \ x > 0, y > 0, \ z > 0.$$

Since the square root is increasing, it is the same as to minimize

$$\min(x^2 + y^2 + z^2) \quad \text{subject to } \frac{1}{xy} + \frac{1}{yz} \leq 1, \ x > 0, \ y > 0, \ z > 0$$

but do not forget to take the square root at the end.

Introduce the open set $X = \{x > 0, \ y > 0, \ z > 0\}$. The set is convex as intersection of three open half-spaces. The objective function $x^2 + y^2 + y^2$ is convex as a sum of three scalar convex functions. The inequality constraint function

$$g(x, y, z) = \frac{1}{xy} + \frac{1}{yz}$$

is a sum of two functions. Let us test if those are convex. Calculate

$$\nabla\left(\frac{1}{xy}\right) = \begin{bmatrix} -\frac{1}{x^2y} \\ -\frac{1}{xy^2} \end{bmatrix}, \qquad H = \nabla^2\left(\frac{1}{xy}\right) = \begin{bmatrix} \frac{2}{x^3y} & \frac{1}{x^2y^2} \\ \frac{1}{x^2y^2} & \frac{2}{xy^3} \end{bmatrix}.$$

Now in X

$$\det H_1 = \frac{2}{x^3y} > 0, \qquad \det H_2 = \frac{4}{x^4y^4} - \frac{1}{x^4y^4} = \frac{3}{x^4y^4} > 0.$$

Thus, by Sylvester the function $\frac{1}{xy}$ is convex in X, and the same is true for $\frac{1}{yz}$ and also for g as being a sum of convex functions.

Therefore, the problem is convex, and if we find a KKT point then it is the global minimum.

Stating the KKT condition to take the chance

$$\begin{cases} 2x - \frac{u}{x^2y} = 0, & \Rightarrow \quad 2x^3y = u & (1) \\ 2y - u\left(\frac{1}{xy^2} + \frac{1}{y^2z}\right) = 0, & \Rightarrow \quad 2y^2 - u\left(\frac{1}{xy} + \frac{1}{yz}\right) = 0 & (2) \\ 2z - \frac{u}{yz^2} = 0, & \Rightarrow \quad 2z^3y = u, & (3) \\ u\left(\frac{1}{xy} + \frac{1}{yz} - 1\right) = 0, & \Rightarrow \quad u\left(\frac{1}{xy} + \frac{1}{yz}\right) = u & (4) \\ \frac{1}{xy} + \frac{1}{yz} \leq 1, \ x, y, z > 0. & \end{cases}$$

From (1) and (3) we have $u = 2x^3y = 2z^3y$ and from (2) and (4) we have $u = 2y^2$. Then

$$u = 2x^3y = 2z^3y = 2y^2 \quad \Rightarrow \quad y = x^3 = z^3, \quad x = z.$$

We also know that $u = y^2 > 0$, hence from (4) we get

$$1 = \frac{1}{xy} + \frac{1}{yz} = [x = z] = \frac{2}{xy} = [y = x^3] = \frac{2}{x^4} \quad \Rightarrow \quad x = \sqrt[4]{2}.$$

It makes $y = \sqrt[4]{2}^3$. We have found a KKT point, which is the global minimum as the problem is convex.

Finally, the distance is

$$\sqrt{x^2 + y^2 + z^2} = \sqrt{\sqrt{2} + 2\sqrt{2} + \sqrt{2}} = \sqrt{4\sqrt{2}} = 2\sqrt[4]{2}.$$

6. a) The Lagrange function is

$$L(x, u) = x^2 + 2xy + 2y^2 + u(1 - 2y - x^2) = (1 - u)x^2 + 2xy + 2y^2 - 2uy + u.$$

If $u > 1$ then taking $y = 0$ and $x \to +\infty$ we get $L \to -\infty$.
If $0 \le u \le 1$ then minimization w.r.t. $x \ge 0$ will be at $x = 0$

$$\min_x(\underbrace{(1 - u)x^2 + 2yx}_{\ge 0} + 2y^2 - 2uy + u) = 2y^2 - 2uy + u.$$

To minimize w.r.t. y we notice that that function is convex in y (the second derivative is $4 > 0$), so if we can find a stationary point then it will be the global minimum. Let us try:

$$4y - 2u = 0 \quad \Leftrightarrow \quad y = \frac{u}{2} \ge 0.$$

Yes, there is a stationary point in the set X, hence, it is the minimum. The dual function is then

$$\Theta(u) = \begin{cases} -\frac{u^2}{2} + u & \text{if } u \in [0, 1], \\ -\infty & \text{otherwise.} \end{cases}$$

To maximize the dual function — it is concave, let's try again a stationary point (or complete the square)

$$\Theta'(u) = -u + 1 = 0 \quad \Rightarrow \quad \bar{u} = 1 \in [0, 1].$$

So the maximum is $\Theta(\bar{u}) = \Theta(1) = \frac{1}{2}$. The candidates on the way were

$$\bar{x} = 0, \qquad \bar{y} = \frac{\bar{u}}{2} = \frac{1}{2}.$$

Let us try these values

$$f(\bar{x}, \bar{y}) = 0 + 0 + 2\left(\frac{1}{2}\right)^2 = \frac{1}{2} = \Theta(\bar{u}).$$

Yes, no duality gap. It makes $\bar{x}$ the global minimum.

Answer: $(x, y) = (0, \frac{1}{2})$.

b) No duality gap implies saddle point (Th. 3, p.298) which implies optimality (Th.1(1), p.296).