

Students may use the formulae sheet and a pocket calculator.

All solutions must be properly justified.

1. a) Prove that the intersection of two convex sets is convex. (0.2)

b) Prove that two consecutive search directions in the Steepest Descent method with the exact line search are orthogonal. (0.3)

c) Prove or disprove the implication (0.5)

$$\begin{cases} x_1 + 2x_2 + x_3 \leq 0, \\ x_1 - x_2 + 2x_3 \leq 0, \\ x_1 + x_2 - x_3 \leq 0 \end{cases} \Rightarrow 3x_1 + 5x_2 + x_3 \leq 0.$$

2. Consider the function $f(x, y) = x^2 + xy + y^2 - x + y$.

a) Find, if it exists, the minimum of f on \mathbb{R}^2 . (0.4)

b) Suggest a suitable penalty function $\mu \cdot \alpha(x, y)$ and apply the penalty method to

$$\min f(x, y) \quad \text{subject to } y \geq 0. \quad (*)$$

Take as an initial point the optimal point in 2a) and perform the penalty minimization analytically for a given μ . Do the optimal points converge when $\mu \rightarrow +\infty$? Is the limit the solution to the constrained optimization problem (*)? (0.6)

3. Consider the following LP problem

$$\min (3x_1 + x_2 + 3x_3) \quad \text{subject to} \quad \begin{cases} x_1 + x_2 + 2x_3 = c_1, \\ 3x_1 + x_2 + x_3 \geq c_2, \\ \text{all } x_k \geq 0. \end{cases}$$

a) Take $(c_1, c_2) = (1, 2)$. State the dual problem, solve it graphically and use the Complementary Slackness Principle to solve the primal problem. (0.6)

b) Find, e.g. graphically, the cone of all possible vectors $c = (c_1, c_2)$ such that the primal solution is the same as for $c = (1, 2)$ in 3a). (0.4)

Please, turn over

4. a) Is the function $f(x, y) = \max\{(x + y)^2, (x + y)^3\}$ convex on \mathbb{R}^2 ? (0.3)

b) Is the function $f(x, y) = 2^x \cdot 4^{y^2}$ convex on \mathbb{R}^2 ? (0.3)

c) Denote

$$H = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & a \end{bmatrix}.$$

Find all $a \in \mathbb{R}$ such that the set

$$\{x \in \mathbb{R}^3 : x^T H x \leq 1\}$$

is convex. (0.4)

5. Solve the optimization problem

$$\min(x^2 + y^2 + z^2 - 4y - 3z) \quad \text{subject to } 0 \leq z \leq x^2 - y^2$$

using the KKT condition.

6. a) Solve the optimization problem in 5 by the duality method with (0.6)

$$X = \{(x, y, z) : z \geq 0\}.$$

b) Prove that if the Hessian is positive definite then the Newton direction is a descent direction. (0.4)

GOOD LUCK!