LUNDS TEKNISKA HÖGSKOLA
MATEMATIK

TENTAMENSSKRIVNING
OPTIMERING
2019-04-29 kl 14-19

Students may use the formulae sheet and a pocket calculator.
Solutions must be properly justified with all details.

1. a) Comparing the Steepest Descent method (SD) and Newton's method (NM): provide an example of a function $f(x, y)$, for which

(i) NM works very well, but SD works badly, and

(ii) SD works very well, but NM works badly. (0.4)

b) Solve the optimization problem

$$\min \|Ax - b\|$$

where

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}. \qquad (0.3)$$

c) Alice is studing Optimization and comes up with an optimization problem she has got trouble with. Here it is

$$\min\ (x^2y + xy^2) \qquad \text{subject to } x + y \geq 1.$$

Do your best to help her to tackle this problem. (0.3)

2. Consider the function

$$f(x, y, z) = 2x^2 + (x + y)^2 + (x + z)^2 + 2ayz.$$

a) For what values of $a \in \mathbb{R}$ is the function convex in $\mathbb{R}^3$? (0.5)

b) Set $a = 0$. Let H be the Hessian of f and $d_1 = (1\ 0\ 0)^T$. Find vectors d_2 and d_3 such that all three vectors are H-conjugate. (0.5)

3. a) Consider the LP problem

$$\min_{x \in S} c^T x, \quad S = \{x \in \mathbb{R}^n \colon Ax \geq b,\ x \geq 0\}.$$

Let T denote the feasible set for the dual problem. Prove that $x \in S$, $y \in T$ satisfy the Complementary Slackness Principle equations if and only if $c^T x = b^T y$. (0.4)

b) Consider the following LP problem

$$\min\,(2x_1 + x_2 + x_3 + x_4) \qquad \text{subject to} \quad \begin{cases} x_1 - 2x_2 + 3x_3 + 2x_4 & \geq & c_1, \\ -x_1 + x_2 - x_3 + 2x_4 & \geq & c_2, \\ x_1 \ \geq \ 0,\ x_2 & \geq & 0, \\ x_3 \ \leq \ 0,\ x_4 & \leq & 0. \end{cases}$$

State the dual problem. Make a graphical argument for solving the dual problem in order to find all (c_1, c_2) such that the primal feasible set is empty. (0.6)

Please, turn over

4. a) Let $f, g\colon \mathbb{R}^n \to \mathbb{R}$ and $\phi\colon \mathbb{R} \to \mathbb{R}$ be given functions. Determine if each statement is true or false (providing a proof or a counter-example):

(i) if f, g are convex then $h(x, y) = (f(x) + g(y))^2$ is convex.

(ii) if f, ϕ are convex and differentiable, and $\phi' > 0$ then $\phi(f(x))$ is convex.

(iii) if f, g are concave then $\min\{f(x), g(x)\}$ is concave. (0.6)

b) Verify if the following set is convex (0.4)

$$S = \{(x, y) \in \mathbb{R}^2 \colon x \geq 0,\ y \geq 0,\ (\sqrt{x} + \sqrt{y})^2 \geq 2\}.$$

5. Use the KKT method to find the shortest distance from the origin to points within the intersection of the two sets in $\mathbb{R}^3$ that are given by $x^2 + y^2 - z^2 \geq 1$ and $x^2 + y^2 + z = 1$, respectively. In particular, determine all KKT and CQ points for the problem.

6. a) Consider the set $X = \{(x, y) \in \mathbb{R}^2 \colon x \geq 0,\ y \geq 0\}$ and the optimization problem

$$\min(x^2 + xy + y^2) \quad \text{subject to } (x, y) \in X,\ xy \geq 2.$$

Calculate the dual function and find a possible saddle point of the Lagrange function. (0.5)

b) Let $f\colon \mathbb{R} \to \mathbb{R}$ be a convex function that is bounded from above, i.e. $f(x) \leq C$ for some $C \in \mathbb{R}$. Prove that f is constant. (0.5)

GOOD LUCK!