LUNDS TEKNISKA HÖGSKOLA
MATEMATIK

TENTAMENSSKRIVNING
OPTIMERING
2019-01-09 kl 08-13

Students may use the formulae sheet and a pocket calculator.
Solutions must be properly justified with all details.

1. a) Let

$$H = \begin{bmatrix} 1 & a & 1 \\ a & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix}.$$

Find all values of $a \in \mathbb{R}$ such that the function $\frac{1}{2}x^T Hx$ is convex on $\mathbb{R}^3$. (0.4)

b) Find all possible values of $a, b, c \in \mathbb{R}$ such that the minimum of the function

$$f(x, y, z) = (x + y + z)^2 + ax + by + cz$$

in $\mathbb{R}^3$ exists (finite). (0.3)

c) Consider a differentiable function $f: \mathbb{R}^n \to \mathbb{R}$ and let $x, d \in \mathbb{R}^n$ be fixed. Prove that if $f(x + \lambda d)$ attains its minimum at λ_* then $\nabla f(x + \lambda_* d) \perp d$. (0.3)

2. Consider the optimization problem

$$\min(x + y) \quad \text{subject to } y \geq x^2 \qquad (*)$$

and let $\epsilon > 0$ and

$$q_\epsilon(x, y) = x + y - \epsilon \ln(y - x^2), \qquad D_q = \{(x, y) : y > x^2\}.$$

a) Is the function q_ϵ convex on D_q?

b) Would it be a proper function to use in the barrier method for the optimization problem (*)?

c) Does the minimum of q_ϵ on D_q exist for every $\epsilon > 0$? If yes, calculate the minimum for every $\epsilon > 0$ and verify whether it converges to the solution of the optimization problem (*) when $\epsilon \to 0^+$ or not.

3. a) Consider the following LP problem

$$\max(x_1 + 4x_2 + 3x_3) \qquad \text{subject to} \quad \begin{cases} x_1 + 3x_2 + x_3 & = & 6, \\ 3x_1 + 5x_2 + x_3 & \leq & 7, \\ 3x_1 + x_2 + x_3 & \geq & 2, \\ \text{all } x_k \geq 0. \end{cases}$$

State the dual problem and verify by the Complementary Slackness Principle that $(0, 0, 6)$ solves the problem above. (0.6)

b) For a $m \times n$ matrix A and a vector b, use Farkas' theorem to prove that the following statements are equivalent:

1. $Ax = b$ has a solution.
2. $A^T y = 0 \Rightarrow b^T y = 0$. (0.4)

Please, turn over

4. a) A function $f\colon \mathbb{R}^n \to \mathbb{R}$ is called *coordinate-wise increasing* if for all $x, y \in \mathbb{R}^n$ it holds

$$x \le y \quad \Rightarrow \quad f(x) \le f(y).$$

Prove that the function $h(z) = f(g_1(z), \ldots, g_n(z))$ is convex if the functions $g_1, \ldots, g_n\colon \mathbb{R}^m \to \mathbb{R}$ are convex and the function $f\colon \mathbb{R}^n \to \mathbb{R}$ is coordinate-wise increasing and convex. (0.3)

b) For each function below, determine whether it is convex or not. (0.4)

$$h_1(x,y,z) = \sqrt{x^6 + y^6 + z^6}, \qquad h_2(x,y,z) = \sqrt[6]{x^2 + y^2 + z^2}.$$

c) Find all $a \in \mathbb{R}$ such that the function $f(x,y) = (x + y^2)^2 + ay^4$ is convex on $\{(x,y)\colon x \ge 0\}$. (0.3)

5. Use the KKT method to solve the optimization problem

$$\min(xy + 2\ln z) \quad \text{subject to } xyz \ge 1,\ x + y = 4,\ x, y, z > 0.$$

6. a) Consider the set $X = \{(x,y) \in \mathbb{R}^2\colon y \ge 0\}$ and the optimization problem

$$\min(x^2 - 12x + y^2 + 2y) \quad \text{subject to } (x,y) \in X,\ x^2 + y \le 4,\ x^2 - y^2 \ge 1.$$

Calculate the dual problem and solve the problem above by duality. (0.6)

b) Consider a general form of the constrained optimization problem

$$\min f(x) \quad \text{subject to } x \in X,\ g(x) \le 0,\ h(x) = 0$$

and its dual function $\Theta(u,v)$. Prove that for all primal feasible x and all dual feasible (u,v) it holds

$$\Theta(u,v) \le f(x).$$

(0.4)

GOOD LUCK!