LUNDS TEKNISKA HÖGSKOLA
MATEMATIK

TENTAMENSSKRIVNING
OPTIMERING
2018-04-11 kl 14-19

Students may use the formulae sheet and a pocket calculator. **All solutions must be properly justified**.

1. a) Consider the matrix

$$H = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Starting with the vector $d_1 = (1, -1, 1)^T$ find vectors d_2, d_3 such that all three vectors are H-conjugate. (0.5)

b) Let f be a quadratic function with a positive-definite Hessian H. Given a point x and a direction $d \neq 0$, prove that the minimum of $\phi(\lambda) = f(x + \lambda d)$ exists and is attained at

$$\lambda_* = -\frac{d^T \nabla f(x)}{d^T H d}.$$

(0.5)

2. In this problem, we take the matrix H from 1a) above.

a) Determine if H is (positive or negative) definite, semidefinite or indefinite. (0.3)

b) Consider the function $f(x) = \frac{1}{2} x^T H x$. Let $\epsilon \in \mathbb{N}$ be the smallest possible natural number such that the matrix $H + \epsilon I$ is positive-definite. Perform 2-3 steps of the Modified Newton method starting at the point $(1, -1, 1)$ to conclude whether you think the method converges. (0.5)

c) Does the minimum of $f(x) = \frac{1}{2} x^T H x$ exist? (0.2)

3. a) Consider the following LP problem

$$\min\,(3x_1 + 6x_2 + 3x_3) \qquad \text{subject to} \qquad \begin{cases} x_1 + 2x_2 + 2x_3 \geq -1, \\ x_1 + x_2 - x_3 = 2, \\ x_1,\ x_2 \geq 0, \\ x_3 \text{ free.} \end{cases}$$

State the dual problem, solve the dual problem graphically and use the Complementary Slackness Principle to solve the problem above. (0.6)

b) Find all vectors $c = (c_1, c_2)$ that satisfy

$$\begin{cases} 2x - y \leq 0, \\ -x + 3y \leq 0 \end{cases} \qquad \Rightarrow \qquad c_1 x + c_2 y \leq 0,$$

and draw them in the plane. (0.4)

Please, turn over

4. a) Determine if the following functions are convex: (0.7)

- $(x+y+z)^3$ in $\mathbb{R}^3$;
- $(x^2+y^2+z^2)^3$ in $\mathbb{R}^3$;
- $\sqrt[3]{x^3+y^3}$ in $\{(x,y)\colon x>0,\ y>0\}$.

b) Is the set

$$\Omega = \{(x_1,x_2,x_3,x_4)\colon 0\le x_1\le 1,\ 0\le x_2\le 1,\ 0\le x_3\le 1,\ x_4\ge x_1^2+x_2^2+x_3^2\}$$

convex? (0.3)

5. Use the necessary KKT condition to solve the optimization problem

$$\min(x+2y) \quad \text{subject to } \sin x \le y \le x.$$

6. a) Solve the optimization problem

$$\min(xy+y^2) \quad \text{subject to } y^2-x^2\ge 1,\ x\ge 0,\ y\ge 0,$$

using the duality method with $X=\{(x,y)\colon x\ge 0,\ y\ge 0\}$. (0.6)

b) Use the definition of a convex function to prove the following result: for a convex C^1 function f on $[a,b]$ and $c\in(a,b)$ it holds

$$f(c)=f'(c)=0 \qquad \Rightarrow \qquad f(x)\ge 0 \text{ for all } x\in[a,b].$$

(0.4)

GOOD LUCK!