LUNDS TEKNISKA HÖGSKOLA
MATEMATIK

TENTAMENSSKRIVNING
OPTIMERING
2018-01-11 kl 08-13

Students may use the formulae sheet and a pocket calculator. **All solutions must be properly justified**.

1. Consider the set

$$S = \{(x, y) \in \mathbb{R}^2 \colon x^2 + (y-1)^2 \le 4,\ x^2 + (y+1)^2 \ge 4,\ x^2 + (y-2)^2 \ge 1\}.$$

a) Draw the set S and find graphically (or by any other means) all the points where the constraint qualification is not satisfied (aka CQ points). (0.4)

b) Take the point $a = (\sqrt{3}, 0) \in S$. Draw the set of all feasible directions at a. Draw the set of all possible gradients $\nabla f(a)$ that satisfy the (geometrical) necessary condition for local minimum in the optimization problem $\min_{x\in S} f(x)$. (Make separate drawings and ensure that the sets can be clearly identified.) (0.6)

2. Consider the optimization problem

$$\min x^2 + 4y^2 \qquad \text{subject to } xy \ge 1.$$

a) Prove that the solution exists and calculate it. Is the problem convex? (0.3)

b) Add a differentiable penalty function with $\mu = 1$. Make 2 steps of Newton's method from the starting point $(1, 2)$. Does the method converge? What happens if you take $\mu = 10$? (0.3)

c) Add a differentiable barrier function with $\epsilon = 1$ and calculate the first Newton direction from the same starting point $(1, 2)$. Is this direction pointing better to the minimum than the first direction in b)? (0.4)

3. a) Consider the following LP problem

$$\max\,(x_1 + 2x_2 + 4x_3) \qquad \text{subject to} \qquad \begin{cases} 5x_1 + 6x_2 + x_3 & \ge & 2, \\ 9x_1 + 3x_2 + 2x_3 & = & 6, \\ 3x_1 - x_2 + x_3 & \le & 4, \\ \text{all } x_k & \ge & 0. \end{cases}$$

Construct the dual problem and verify by the Complementarty Slackness Principle that $(0, 0, 3)$ is the solution to the primal problem. (0.5)

b) Prove, using Farkas' theorem, that exactly one of the following two systems has a solution: (0.5)

- $Ax \le b$,
- $A^T y = 0$, $y \ge 0$, $b^T y < 0$.

Please, turn over

4. a) Prove that the intersection of two convex sets is convex. (0.2)

b) Which of the following functions are convex in $\mathbb{R}^3$? (0.5)

- $(1+(x+y+z)^2)\ln(1+(x+y+z)^2)$,
- $(x+y)(y+z)(x+z)$.

c) Is the set

$$\{(x,y)\colon xy \geq x+y, x>0, y>0\}$$

convex? (0.3)

5. Solve the following minimization problem using the KKT method

$$\min \frac{x^2+y^2}{x+y} \qquad \text{subject to } x>0,\ y>0,\ xy\geq 1.$$

6. a) Problem 5 can be rewritten as

$$\min z^2 \qquad \text{subject to } x^2+y^2 \leq z(x+y),\ xy\geq 1,\ x\geq 0,\ y\geq 0.$$

Solve the problem by the duality method with $X=\{(x,y)\colon x\geq 0,\ y\geq 0\}$. (0.6)
(Hint: do minimization on z first.)

b) Let A be an $m\times n$ real matrix of rank n, and $b\in\mathbb{R}^m$. Prove that x_0 is a solution to

$$\min_{x\in\mathbb{R}^n} \|Ax-b\|^2$$

if and only if x_0 solves the normal equation. (0.4)

GOOD LUCK!