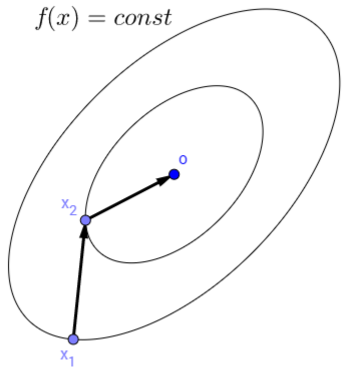
LUNDS TEKNISKA HÖGSKOLA
MATEMATIK

TENTAMENSSKRIVNING
OPTIMERING
2017-01-12 kl 08-13

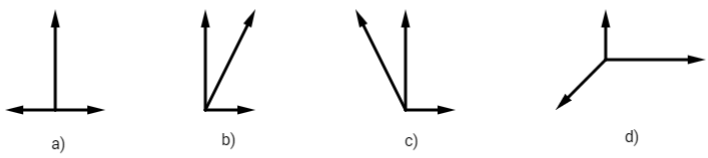
Students may use the formulae sheet and a pocket calculator. **All solutions must be properly justified**.

1. a) Two consecutive iterations of a minimization algorithm are shown in the picture together with some level sets of a positive definite quadratic form $f(x) = x^T Hx$, $x \in \mathbb{R}^2$. Explain for each method below why it could or could not be the one used (with exact line search): (0.6)



- Steepest descent method,
- Conjugate directions method,
- Newton's method,
- Quasi-Newton method.

b) Assume that the vectors below are gradients of active inequality constraints at a point in the plane. In what case(s) is the point a CQ point? (0.4)



2. Consider the function $f(x, y) = x^3 - 3xy + y^3$ on the set $S = \{(x, y)\colon x \geq 0, xy \geq 1\}$.

a) Show that f is convex on S. (0.3)

b) Show that the minimum of f on S exists. (0.3)

c) Construct a differentiable penalty function for the problem $\min_{x \in S} f(x)$ and find the first Newton direction starting at $(1, 0)$ with $\mu = 3$. (0.4)

3. a) Consider the following LP problem

$$\min\,(9x_1 + 6x_2 + 8x_3 - 2x_4) \qquad \text{subject to} \qquad \begin{cases} 3x_1 + x_2 - x_3 - 2x_4 & \geq & 1, \\ x_1 - x_2 - 2x_3 + x_4 & \leq & 1, \\ \text{all } x_k & \geq & 0. \end{cases}$$

Construct the dual problem, solve it graphically and use the solution to solve the primal problem. (0.6)

Please, turn over

b) Let vectors b, c and a matrix A (of suitable dimensions) be given and consider two LP problems

$$P_1 : \min c^T x \text{ subject to } \begin{cases} Ax \geq b \\ x \geq 0 \end{cases} \quad \text{and} \quad P_2 : \min c^T x \text{ subject to } \begin{cases} \begin{bmatrix} A \\ I \end{bmatrix} x \geq \begin{bmatrix} b \\ 0 \end{bmatrix}, \\ x \text{ is free.} \end{cases}$$

They are the same problem, but in P_2 the positivity constraints are considered to be a part of inequalities within the larger matrix $\begin{bmatrix} A \\ I \end{bmatrix}$. Formulate the dual problems to P_1 and P_2 and show that the dual problems are equivalent. (0.4)

4. a) Let A be a $m \times n$ matrix and $b \in \mathbb{R}^m$. Prove that $h(x) = Ax + b$ is convex. (0.2)

b) Is the set

$$\Omega = \{(x, y, z) \colon \frac{x^2}{y} + \frac{y^2}{z} + \frac{z^2}{x} \leq 1, \ x > 0, \ y > 0, \ z > 0\}$$

convex? (0.4)

c) Let $g \colon \mathbb{R}^n \to \mathbb{R}$ be a convex function. Show that the penalty function

$$\alpha(x) = \max\{g(x), 0\}^2$$

is convex too. (0.4)

5. Find the (minimal) distance (if it exists) from the origin to the set

$$S = \left\{(x, y, z) \colon x > 0, \ y > 0, \ z > 0, \ \frac{1}{xy} + \frac{1}{yz} \leq 1\right\}.$$

6. a) Solve the optimization problem

$$\min(x^2 + 2xy + 2y^2) \quad \text{subject to } 2y + x^2 \geq 1, \ x \geq 0, \ y \geq 0$$

by the duality method with $X = \{(x, y) \colon x \geq 0, \ y \geq 0\}$. (0.5)
(Hint: do minimization in x first.)

b) Consider a primal problem

$$\min f(x) \quad \text{subject to } g(x) \leq 0, \ h(x) = 0$$

where g and h are vector valued functions. Prove the if there exist a primal feasible $\bar{x}$ and a dual feasible $(\bar{u}, \bar{v})$ such that

$$f(\bar{x}) = \Theta(\bar{u}, \bar{v})$$

then $\bar{x}$ is the global minimum for the primal problem. (0.5)

GOOD LUCK!