Lund University

Centre for Mathematical Sciences
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Linear Algebra 2
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Duration: 08:00–13:00

Solutions

1. Let

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} 3 \\ 5 \\ 8 \\ 10 \end{bmatrix}.$$

Then

$$A^tA = \begin{bmatrix} 14 & 6 \\ 6 & 4 \end{bmatrix} \quad \text{and} \quad A^tY = \begin{bmatrix} 51 \\ 26 \end{bmatrix},$$

and the solution of the normal equations

$$A^tA \begin{bmatrix} a \\ b \end{bmatrix} = A^tY$$

is given by

$$a = \frac{12}{5}, \quad b = \frac{29}{10}.$$

Answer: $y = \frac{12}{5}t + \frac{29}{10}$.

2. The eigenvalues of A are $\lambda_1 = 2$ and $\lambda_2 = 7$. The corresponding eigenvectors are the non-zero vectors given by $\boldsymbol{x} = t\boldsymbol{e}_1$ and $\boldsymbol{x} = t\boldsymbol{e}_2$, respectively, where $\boldsymbol{e}_1 = (3, -2)$ and $\boldsymbol{e}_2 = (1, 1)$. We see that $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ form a basis for $\mathbf{R}^2$. Hence, A is diagonalisable, and $A = TDT^{-1}$ where

$$T = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix}.$$

The inverse of T is

$$T^{-1} = \frac{1}{5}\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}.$$

Consequently,

$$\begin{aligned} A^n = \left(TDT^{-1}\right)^n = TD^nT^{-1} &= \frac{1}{5}\begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 2^n & 0 \\ 0 & 7^n \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \\ &= \frac{1}{5}\begin{bmatrix} 3 \cdot 2^n + 2 \cdot 7^n & -3 \cdot 2^n + 3 \cdot 7^n \\ -2 \cdot 2^n + 2 \cdot 7^n & 2 \cdot 2^n + 3 \cdot 7^n \end{bmatrix}. \end{aligned}$$

Answer: $A^n = \frac{1}{5}\begin{bmatrix} 3 \cdot 2^n + 2 \cdot 7^n & -3 \cdot 2^n + 3 \cdot 7^n \\ -2 \cdot 2^n + 2 \cdot 7^n & 2 \cdot 2^n + 3 \cdot 7^n \end{bmatrix}$, $n \in \mathbf{Z}_+$.

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3. The matrix of the quadratic form is

$$B = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 3 & 2 \\ 2 & 2 & 6 \end{bmatrix},$$

and

$$\begin{aligned} \det(B - \lambda I) &= \begin{vmatrix} 3-\lambda & 1 & 2 \\ 1 & 3-\lambda & 2 \\ 2 & 2 & 6-\lambda \end{vmatrix} = \begin{vmatrix} 3-\lambda & 1 & 2 \\ \lambda - 2 & 2-\lambda & 0 \\ 2 & 2 & 6-\lambda \end{vmatrix} \\ &= \begin{vmatrix} 3-\lambda & 4-\lambda & 2 \\ \lambda - 2 & 0 & 0 \\ 2 & 4 & 6-\lambda \end{vmatrix} = -(\lambda-2)(\lambda^2 - 10\lambda + 16) = -(\lambda-2)^2(\lambda-8). \end{aligned}$$

Hence, the eigenvalues of the quadratic form are 2 and 8. Since the eigenvalue 2 occurs twice, the surface is a surface of revolution. Since the eigenvalues and the right-hand side of the equation are positive, the surface is an ellipsoid. An eigenvector belonging to the eigenvalue 8 is $(1, 1, 2)$. Therefore, the axis of revolution is the line through the origin spanned by the vector $(1, 1, 2)$.

Answer: The surface is an ellipsoid. The axis of revolution is the line through the origin spanned by the vector $(1, 1, 2)$.

4. Let A be the matrix. We have that F is an orthogonal projection if and only if F is symmetric and $F^2 = F$. Since the basis is orthonormal, this is equivalent to $A^t = A$ and $A^2 = A$. The first condition means that $a = 1$. The entry of A^2 in position $(1, 2)$ is

$$\left(\frac{1}{6}\right)^2 (5 \cdot 1 + 1 \cdot b + 2 \cdot (-2)) = \frac{1+b}{36}.$$

In order that $A^2 = A$, we must therefore have that

$$\frac{1+b}{36} = \frac{1}{6} \quad \Leftrightarrow \quad b = 5.$$

Hence, the only possible solution is $a = 1$, $b = 5$. For these values of a and b, we have that

$$A^2 = \frac{1}{36} \begin{bmatrix} 30 & 6 & 12 \\ 6 & 30 & -12 \\ 12 & -12 & 12 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 5 & 1 & 2 \\ 1 & 5 & -2 \\ 2 & -2 & 2 \end{bmatrix} = A.$$

Consequently, F is an orthogonal projection for $a = 1$ and $b = 5$. The subspace onto which the projection is made is given by

$$(A - I)\boldsymbol{x} = \mathbf{0} \quad \Leftrightarrow \quad \left[\begin{array}{ccc|c} -1 & 1 & 2 & 0 \\ 1 & -1 & -2 & 0 \\ 2 & -2 & -4 & 0 \end{array}\right] \quad \Leftrightarrow \quad x_1 - x_2 - 2x_3 = 0.$$

Answer: $a = 1$, $b = 5$ and the subspace onto which the projection is made is given by $x_1 - x_2 - 2x_3 = 0$.

5. Confer Definition 6.19 and Theorem 6.20.

6. We show that $A+I$ is invertible by showing that the only solution of the equation $(A+I)\boldsymbol{x}=\mathbf{0}$ is $\boldsymbol{x}=\mathbf{0}$. If $(A+I)\boldsymbol{x}=\mathbf{0}$, then $\boldsymbol{x}^t(A+I)\boldsymbol{x}=\boldsymbol{x}^t\mathbf{0}=0$. It then follows that $\boldsymbol{x}^t(A^t+I)\boldsymbol{x}=(\boldsymbol{x}^t(A+I)\boldsymbol{x})^t=0^t=0$, and since A is skew-symmetric, we obtain that $\boldsymbol{x}^t(-A+I)\boldsymbol{x}=0$. Hence, $2\|\boldsymbol{x}\|^2=\boldsymbol{x}^t(2I)\boldsymbol{x}=\boldsymbol{x}^t(A+I)\boldsymbol{x}+\boldsymbol{x}^t(-A+I)\boldsymbol{x}=0$, from which it follows that $\boldsymbol{x}=\mathbf{0}$. This concludes the proof.