LUND UNIVERSITY

Linear Algebra 2
Thursday, 27 October 2016
Duration: 14:00–19:00

Centre for Mathematical Sciences
Mathematics, Faculty of Science

Solutions

1. Set

$$A = \begin{bmatrix} 1 & 2 & -3 & -7 \\ 2 & 5 & -7 & -15 \end{bmatrix}.$$

Then $\boldsymbol{x} \in \ker A$ if and only if

$$A\boldsymbol{x} = \boldsymbol{0} \quad \Leftrightarrow \quad \left[\begin{array}{cccc|c} 1 & 2 & -3 & -7 & 0 \\ 2 & 5 & -7 & -15 & 0 \end{array}\right] \quad \Leftrightarrow \quad \left[\begin{array}{cccc|c} 1 & 2 & -3 & -7 & 0 \\ 0 & 1 & -1 & -1 & 0 \end{array}\right]$$
$$\Leftrightarrow \quad \boldsymbol{x} = s(1,1,1,0) + t(5,1,0,1)$$

for some real numbers s and t. Hence, the vectors $\boldsymbol{v}_1 = (1,1,1,0)$ and $\boldsymbol{v}_2 = (5,1,0,1)$ span $\ker A$, and since they are linearly independent, they form a basis for $\ker A$. We now apply the Gram–Schmidt process to them to obtain an orthonormal basis for $\ker A$. Set $\boldsymbol{u}_1 = \boldsymbol{v}_1$ and $\boldsymbol{u}_2 = s\boldsymbol{u}_1 + \boldsymbol{v}_2$. Then $\langle \boldsymbol{u}_1, \boldsymbol{u}_2 \rangle = 0$ if

$$s = -\frac{\langle \boldsymbol{u}_1, \boldsymbol{v}_2 \rangle}{\|\boldsymbol{u}_1\|^2} = -\frac{6}{3} = -2.$$

Hence, $\boldsymbol{u}_2 = -2\boldsymbol{u}_1 + \boldsymbol{v}_2 = (3,-1,-2,1)$ is orthogonal to $\boldsymbol{u}_1$ and

$$\boldsymbol{e}_1 = \frac{1}{\|\boldsymbol{u}_1\|}\boldsymbol{u}_1 = \frac{1}{\sqrt{3}}(1,1,1,0) \quad \text{and} \quad \boldsymbol{e}_2 = \frac{1}{\|\boldsymbol{u}_2\|}\boldsymbol{u}_2 = \frac{1}{\sqrt{15}}(3,-1,-2,1)$$

form an orthonormal basis for $\ker A$.

Answer: E.g. $\frac{1}{\sqrt{3}}(1,1,1,0)$, $\frac{1}{\sqrt{15}}(3,-1,-2,1)$.

2. The eigenvalues of the coefficient matrix are given by

$$0 = \begin{vmatrix} 1-\lambda & 2 & 1 \\ 2 & 1-\lambda & -1 \\ -1 & -1 & 1-\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & 1+\lambda & 1 \\ 2 & -1-\lambda & -1 \\ -1 & 0 & 1-\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & 1+\lambda & 1 \\ 3-\lambda & 0 & 0 \\ -1 & 0 & 1-\lambda \end{vmatrix}$$
$$= -(\lambda+1)(\lambda-1)(\lambda-3)$$

and are $\lambda_1 = -1$, $\lambda_2 = 1$ and $\lambda_3 = 3$.
The eigenvectors belonging to λ_1 are the non-zero vectors given by

$$\left[\begin{array}{ccc|c} 2 & 2 & 1 & 0 \\ 2 & 2 & -1 & 0 \\ -1 & -1 & 2 & 0 \end{array}\right] \quad \Leftrightarrow \quad \left[\begin{array}{ccc|c} 0 & 0 & 2 & 0 \\ 0 & 0 & 3 & 0 \\ -1 & -1 & 2 & 0 \end{array}\right] \quad \Leftrightarrow \quad \boldsymbol{x} = t\boldsymbol{e}_1$$

where $\boldsymbol{e}_1 = (1,-1,0)$.

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The eigenvectors belonging to λ_2 are the non-zero vectors given by

$$\left[\begin{array}{ccc|c} 0 & 2 & 1 & 0 \\ 2 & 0 & -1 & 0 \\ -1 & -1 & 0 & 0 \end{array}\right] \quad \Leftrightarrow \quad \left[\begin{array}{ccc|c} 0 & 2 & 1 & 0 \\ 2 & 2 & 0 & 0 \\ -1 & -1 & 0 & 0 \end{array}\right] \quad \Leftrightarrow \quad \boldsymbol{x} = t\boldsymbol{e}_2$$

where $\boldsymbol{e}_2 = (1, -1, 2)$.
Those belonging to λ_3 are the non-zero vectors given by

$$\left[\begin{array}{ccc|c} -2 & 2 & 1 & 0 \\ 2 & -2 & -1 & 0 \\ -1 & -1 & -2 & 0 \end{array}\right] \quad \Leftrightarrow \quad \left[\begin{array}{ccc|c} -4 & 0 & -3 & 0 \\ -1 & -1 & -2 & 0 \end{array}\right] \quad \Leftrightarrow \quad \boldsymbol{x} = t\boldsymbol{e}_3$$

where $\boldsymbol{e}_3 = (3, 5, -4)$.
Since the eigenvalues are distinct, the eigenvectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ form a basis for $\mathbf{R}^3$. The general solution of the system of linear equations is therefore given by

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = c_1 e^{-t} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + c_2 e^{t} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + c_3 e^{3t} \begin{bmatrix} 3 \\ 5 \\ -4 \end{bmatrix}.$$

The initial condition gives

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ -1 & -1 & 5 & 15 \\ 0 & 2 & -4 & -12 \end{array}\right] \quad \Leftrightarrow \quad \left[\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 0 & 0 & 8 & 16 \\ 0 & 2 & -4 & -12 \end{array}\right] \quad \Leftrightarrow \quad \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \\ 2 \end{bmatrix}.$$

Answer: $$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = -3e^{-t} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - 2e^{t} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + 2e^{3t} \begin{bmatrix} 3 \\ 5 \\ -4 \end{bmatrix}.$$

3. The matrix of the quadratic form q is

$$B = \begin{bmatrix} 6 & -2 & 2 \\ -2 & -1 & 0 \\ 2 & 0 & -1 \end{bmatrix},$$

and since

$$\begin{aligned} \det(B - \lambda I) &= \begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & -1-\lambda & 0 \\ 2 & 0 & -1-\lambda \end{vmatrix} = \begin{vmatrix} 6-\lambda & -2 & 0 \\ -2 & -1-\lambda & -1-\lambda \\ 2 & 0 & -1-\lambda \end{vmatrix} \\ &= \begin{vmatrix} 6-\lambda & -2 & 0 \\ -4 & -1-\lambda & 0 \\ 2 & 0 & -1-\lambda \end{vmatrix} = -(\lambda+2)(\lambda+1)(\lambda-7), \end{aligned}$$

the eigenvalues of q are $\lambda_1 = -2$, $\lambda_2 = -1$, $\lambda_3 = 7$. Two eigenvalues are negative, one is positive and the right-hand side of the equation is positive. The surface is therefore a hyperboloid of two sheets. Let $\boldsymbol{e}_1'$, $\boldsymbol{e}_2'$, $\boldsymbol{e}_3'$ be an orthonormal basis of eigenvectors belonging to λ_1, λ_2, λ_3. Then

$$q(\boldsymbol{x}) = -2(x_1')^2 - (x_2')^2 + 7(x_3')^2$$

if $\boldsymbol{x} = x_1'\boldsymbol{e}_1' + x_2'\boldsymbol{e}_2' + x_3'\boldsymbol{e}_3'$. For a point $\boldsymbol{x}$ on the surface, we have that

$$\|\boldsymbol{x}\|^2 = \|\boldsymbol{x}'\|^2 = \frac{1}{7}(7(x_1')^2 + 7(x_2')^2 + 7(x_3')^2) \geq \frac{1}{7}(-2(x_1')^2 - (x_2')^2 + 7(x_3')^2) = \frac{28}{7} = 4$$

with equality if and only if $x_1' = x_2' = 0$ and $x_3' = \pm 2$. Since

$$\begin{aligned} B\boldsymbol{x} = \lambda_3\boldsymbol{x} \quad &\Leftrightarrow \quad \left[\begin{array}{ccc|c} -1 & -2 & 2 & 0 \\ -2 & -8 & 0 & 0 \\ 2 & 0 & -8 & 0 \end{array}\right] \quad \Leftrightarrow \quad \left[\begin{array}{ccc|c} -1 & -2 & 2 & 0 \\ -2 & -8 & 0 & 0 \\ -2 & -8 & 0 & 0 \end{array}\right] \\ &\Leftrightarrow \quad \boldsymbol{x} = t(4,-1,1), \end{aligned}$$

we can take $\boldsymbol{e}_3' = \frac{1}{\sqrt{18}}(4,-1,1)$. The points on the surface closest to the origin are therefore

$$\pm 2\boldsymbol{e}_3' = \pm\frac{\sqrt{2}}{3}(4,-1,1).$$

Answer: The surface is a hyperboloid of two sheets and the points on the surface closest to the origin are $\pm\frac{\sqrt{2}}{3}(4,-1,1)$.

4. If A is the matrix of F, we have that

$$\begin{aligned} \langle A_1, A_1\rangle &= \frac{1}{49}(36+4+9) = 1, \\ \langle A_2, A_2\rangle &= \frac{1}{49}(4+9+36) = 1, \\ \langle A_3, A_3\rangle &= \frac{1}{49}(9+36+4) = 1, \\ \langle A_1, A_2\rangle &= \frac{1}{49}(12+6-18) = 0, \\ \langle A_1, A_3\rangle &= \frac{1}{49}(-18+12+6) = 0, \\ \langle A_2, A_3\rangle &= \frac{1}{49}(-6+18-12) = 0. \end{aligned}$$

Hence, A is orthogonal, and since the basis is orthonormal, F is an isometry. From

$$A\boldsymbol{x} = \boldsymbol{x} \quad \Leftrightarrow \quad \left[\begin{array}{ccc|c} -1 & 2 & -3 & 0 \\ 2 & -4 & 6 & 0 \\ 3 & -6 & -5 & 0 \end{array}\right] \quad \Leftrightarrow \quad \left[\begin{array}{ccc|c} -1 & 2 & -3 & 0 \\ 0 & 0 & -14 & 0 \end{array}\right] \quad \Leftrightarrow \quad \boldsymbol{x} = t(2,1,0),$$

we conclude that $\ker(A - I)$ is the line spanned by the vector $\boldsymbol{w}$ with coordinates $(2,1,0)$. This shows that F is a rotation about that line. Take $\boldsymbol{u}$ to be the vector with coordinates $(1,-2,0)$. Then $\boldsymbol{u}$ is orthogonal to $\boldsymbol{w}$. The coordinates of $\boldsymbol{v} = F(\boldsymbol{u})$ are

$$A\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 2 \\ -4 \\ 15 \end{bmatrix}$$

The angle of rotation θ is the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$, and we have that

$$\cos\theta = \frac{\langle \boldsymbol{u}, \boldsymbol{v}\rangle}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|} = \frac{10}{\sqrt{245}\sqrt{5}} = \frac{10}{35} = \frac{2}{7}.$$

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The determinant of the matrix whose columns are the coordinates of $\boldsymbol{u}$, $7\boldsymbol{v}$ and $\boldsymbol{w}$ is

$$\begin{vmatrix} 1 & 2 & 2 \\ -2 & -4 & 1 \\ 0 & 15 & 0 \end{vmatrix} = -15 \begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix} = -75 < 0.$$

This shows that the rotation appears to be clockwise when looking from the point with coordinates $(2, 1, 0)$ towards the origin.

Answer: F is rotation about the line spanned by the vector with coordinates $(2, 1, 0)$ through the angle θ where $\cos\theta = \frac{2}{7}$ in the clockwise direction when looking from the point with coordinates $(2, 1, 0)$ towards the origin.

5. See the proof of Theorem 3.29.

6. **a)** We have that

$$\begin{aligned} \boldsymbol{x} \in V_\lambda \quad &\Rightarrow \quad A\boldsymbol{x} = \lambda\boldsymbol{x} \\ &\Rightarrow \quad A(B\boldsymbol{x}) = (AB)\boldsymbol{x} = (BA)\boldsymbol{x} = B(A\boldsymbol{x}) = B(\lambda\boldsymbol{x}) = \lambda(B\boldsymbol{x}) \\ &\Rightarrow \quad B\boldsymbol{x} \in V_\lambda. \end{aligned}$$

This proves the assertion.

b) By the spectral theorem, $\mathbf{R}^n$ has a basis consisting of eigenvectors of A. Let $\lambda_1, \ldots, \lambda_k$ be the distinct eigenvalues of A. Then every vector $\boldsymbol{u} \in \mathbf{R}^n$ can be written uniquely as

$$\boldsymbol{u} = \boldsymbol{u}_1 + \cdots + \boldsymbol{u}_k$$

where $\boldsymbol{u}_i \in V_{\lambda_i}$ for $i = 1, \ldots, k$. By the result shown in 6 a), the restriction of the mapping $\boldsymbol{x} \mapsto B\boldsymbol{x}$ to V_{λ_i} is a symmetric linear transformation on V_{λ_i}. Hence, each V_{λ_i} has a basis consisting of eigenvectors of B. These vectors are also eigenvectors of A, and $\boldsymbol{u}_i$ can be written uniquely as a linear combination of them. Consequently, $\boldsymbol{u}$ can be written uniquely as a linear combination of all these eigenvectors taken together, which therefore form a basis for $\mathbf{R}^n$.