LUND UNIVERSITY

Centre for Mathematical Sciences
Mathematics, Faculty of Science

Linear Algebra 2
Thursday, 17 March 2016
Duration: 08:00–13:00

Solutions

1. We must solve, in the sense of least squares, the system of equations

$$\begin{cases} b = 1 \\ a + b = 2 \\ 2a + b = 2 \\ 3a + b = 3 \end{cases}.$$

With

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 3 \end{bmatrix},$$

the solution of the normal equations

$$A^t A \begin{bmatrix} a \\ b \end{bmatrix} = A^t Y \quad \Leftrightarrow \quad \begin{bmatrix} 14 & 6 \\ 6 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 15 \\ 8 \end{bmatrix}$$

is $a = \frac{3}{5}$, $b = \frac{11}{10}$. Hence, the polynomial we seek is $y = \frac{3}{5}t + \frac{11}{10}$.

2. The eigenvalues of the coefficient matrix are given by

$$\begin{vmatrix} 1-\lambda & 2 \\ 5 & 4-\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & 2 \\ 6-\lambda & 6-\lambda \end{vmatrix} = (\lambda+1)(\lambda-6) = 0$$

and are $\lambda_1 = -1$ and $\lambda_2 = 6$. The eigenvectors are the non-zero vectors given by

$$\begin{bmatrix} 2 & 2 \\ 5 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Leftrightarrow \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

and

$$\begin{bmatrix} -5 & 2 \\ 5 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Leftrightarrow \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t \begin{bmatrix} 2 \\ 5 \end{bmatrix},$$

respectively. We can choose the eigenvectors $\boldsymbol{e}_1 = (1, -1), \boldsymbol{e}_2 = (2, 5)$ as a basis for $\mathbf{R}^2$. Hence, the general solution is given by

$$\begin{bmatrix} a_n \\ b_n \end{bmatrix} = c_1(-1)^n \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 6^n \begin{bmatrix} 2 \\ 5 \end{bmatrix}.$$

The initial conditions now give that

$$\begin{bmatrix} 1 & 2 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 20 \end{bmatrix} \quad \Leftrightarrow \quad \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \end{bmatrix},$$

from which it follows that

$$\begin{bmatrix} a_n \\ b_n \end{bmatrix} = -5(-1)^n \begin{bmatrix} 1 \\ -1 \end{bmatrix} + 3 \cdot 6^n \begin{bmatrix} 2 \\ 5 \end{bmatrix}.$$

Please, turn over!

3. The matrix of the quadratic form is

$$B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \end{bmatrix},$$

and

$$\begin{aligned} \det(B - \lambda I) &= \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & -1-\lambda & -1 \\ 1 & -1 & 1-\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & -1-\lambda & -1 \\ 2-\lambda & 0 & 2-\lambda \end{vmatrix} \\ &= \begin{vmatrix} 1-\lambda & 1 & \lambda \\ 1 & -1-\lambda & -2 \\ 2-\lambda & 0 & 0 \end{vmatrix} = -(\lambda+2)(\lambda-1)(\lambda-2). \end{aligned}$$

Hence, the eigenvalues of the form are -2, 1 and 2. Since two eigenvalues are positive, one is negative and the right-hand side is positive, the surface is a hyperboloid of one sheet. With respect to some orthonormal basis consisting of eigenvectors of B, the equation can now be written as

$$-2(x_1')^2 + (x_2')^2 + 2(x_3')^2 = 4.$$

Let $\boldsymbol{x}' = (x_1', x_2', x_3')$ be the coordinates of a point $\boldsymbol{u}$ on the surface with respect to the eigenvector basis. Since this basis is orthonormal, we have that

$$\begin{aligned} \|\boldsymbol{u}\|^2 = \|\boldsymbol{x}'\|^2 = (x_1')^2 + (x_2')^2 + (x_3')^2 &= \frac{1}{2}(2(x_1')^2 + 2(x_2')^2 + 2(x_3')^2) \\ \geq \frac{1}{2}(-2(x_1')^2 + (x_2')^2 + 2(x_3')^2) &= \frac{1}{2} \cdot 4 = 2 \end{aligned}$$

with equality if $(x_1', x_2', x_3') = (0, 0, \pm\sqrt{2})$. Hence, the least distance is $\sqrt{2}$.

4. The matrix A is orthogonal since $A^tA = I$. Hence, F is an isometry. We have that

$$\boldsymbol{x} \in \ker(A - I) \quad \Leftrightarrow \quad \begin{bmatrix} -5 & -4 & -7 \\ 8 & -8 & 4 \\ -1 & -8 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Leftrightarrow \quad \boldsymbol{x} = t(2, 1, -2).$$

Thus $\ker(F - I)$ is one-dimensional and spanned by the vector $\boldsymbol{w}$ with coordinates $\boldsymbol{z} = (2, 1, -2)$. This shows that F is a rotation about the line spanned by $\boldsymbol{w}$. The vector $\boldsymbol{u}$ with coordinates $\boldsymbol{x} = (1, 0, 1)$ is clearly orthogonal to $\boldsymbol{w}$, and the coordinates of $\boldsymbol{v} = F(\boldsymbol{u})$ are $\boldsymbol{y} = A\boldsymbol{x} = \frac{1}{3}(-1, 4, 1)$. We see that $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal. Hence, the angle of rotation is $\frac{\pi}{2}$. Since

$$\begin{vmatrix} 1 & -1 & 2 \\ 0 & 4 & 1 \\ 1 & 1 & -2 \end{vmatrix} = -18 < 0,$$

the vectors $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ are negatively oriented. Therefore, the rotation appears clockwise on looking from the terminal point of $\boldsymbol{w}$ towards the origin.

5. See the proof of Theorem 4.19.

6. **a)** We have that

$$\begin{aligned}\langle a\boldsymbol{f}+b\boldsymbol{g},\boldsymbol{h}\rangle &= \sum_{k=-1}^{1}(a\boldsymbol{f}+b\boldsymbol{g})(k)\boldsymbol{h}(k) = \sum_{k=-1}^{1}(a\boldsymbol{f}(k)+b\boldsymbol{g}(k))\boldsymbol{h}(k) \\ &= \sum_{k=-1}^{1}(a\boldsymbol{f}(k)\boldsymbol{h}(k)+b\boldsymbol{g}(k)\boldsymbol{h}(k)) = a\sum_{k=-1}^{1}\boldsymbol{f}(k)\boldsymbol{h}(k)+b\sum_{k=-1}^{1}\boldsymbol{g}(k)\boldsymbol{h}(k) \\ &= a\langle \boldsymbol{f},\boldsymbol{h}\rangle + b\langle \boldsymbol{g},\boldsymbol{h}\rangle,\end{aligned}$$

which establishes linearity in the first argument,

$$\langle \boldsymbol{f},\boldsymbol{g}\rangle = \sum_{k=-1}^{1}\boldsymbol{f}(k)\boldsymbol{g}(k) = \sum_{k=-1}^{1}\boldsymbol{g}(k)\boldsymbol{f}(k) = \langle \boldsymbol{g},\boldsymbol{f}\rangle,$$

which shows symmetry, and

$$\langle \boldsymbol{f},\boldsymbol{f}\rangle = \sum_{k=-1}^{1}\boldsymbol{f}(k)\boldsymbol{f}(k) = \sum_{k=-1}^{1}(\boldsymbol{f}(k))^2 \geq 0$$

with equality if and only if $\boldsymbol{f}(-1)=\boldsymbol{f}(0)=\boldsymbol{f}(1)=0$. Since the degree of $\boldsymbol{f}$ is at most 2, this can happen only if $\boldsymbol{f}$ is the zero polynomial. This concludes the proof.

b) Since $\langle 1,t\rangle = 1\cdot(-1)+1\cdot 0+1\cdot 1=0$, we have that $\boldsymbol{f}_1=1$ and $\boldsymbol{f}_2=t$ are orthogonal. Set $\boldsymbol{f}_3=a\cdot 1+b\cdot t+t^2$. We have that $\langle 1,1\rangle=3$, $\langle 1,t^2\rangle=2$, $\langle t,t\rangle=2$ and $\langle t,t^2\rangle=0$. Hence, $\boldsymbol{f}_3$ is orthogonal to $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$ if $3a+2=0$ and $2b+0=0$, that is if

$$\boldsymbol{f}_3 = -\frac{2}{3}+t^2 = \frac{3t^2-2}{3}.$$

We replace $\boldsymbol{f}_3$ with $3t^2-2$. We have seen that $\|\boldsymbol{f}_1\|^2=3$, $\|\boldsymbol{f}_2\|^2=2$, and we have that

$$\|\boldsymbol{f}_3\|^2 = (3-2)^2+(-2)^2+(3-2)^2 = 6.$$

We can therefore choose

$$\boldsymbol{e}_1=\frac{1}{\sqrt{3}},\quad \boldsymbol{e}_2=\frac{1}{\sqrt{2}}t,\quad \boldsymbol{e}_3=\frac{1}{\sqrt{6}}(3t^2-2)$$

as an orthonormal basis for P_2 with this inner product.