SOLUTIONS TO THE EXAM 170318

1. It is clear that f must assume its largest value in the interior of the triangle $x + y \leq 3$, $x, y \geq 0$, because f vanishes on the boundary and is positive inside and negative outside that triangle. The only interior critical point is $(1,1)$ and hence the maximum must be $f(1,1) = 1$.

2. Since the integrand is non-negative, the generalized integral makes sense as either a real number or $+\infty$, and we can use iterated integration to evaluate it. In space-polar coordinates, the domain corresponds to

$$\tilde{D} : r \geq 0,\ 0 \leq \theta < 2\pi,\ 0 \leq \phi \leq \pi/6,$$

and so the integral becomes

$$\begin{aligned}
&\iiint_{\tilde{D}} \frac{r^2 \cos^2\theta \sin^2\phi \cdot r\cos\phi}{r^2} \cdot e^{-r^2} \cdot r^2 \sin\phi \, dr d\phi d\theta \\
&= \int_0^\infty r^3 e^{-r^2}\, dr \int_0^{2\pi} \cos^2\theta \, d\theta \int_0^{\pi/6} \sin^3\phi \cos\phi \, d\phi \\
&= \frac{1}{2}\int_0^\infty t e^{-t}\, dt \cdot \int_0^{2\pi} \frac{1+\cos 2\theta}{2}\, d\theta \cdot \left[\frac{\sin^4\phi}{4}\right]_0^{\pi/6} \\
&= \frac{1}{2} \cdot \pi \cdot \frac{1}{64} = \frac{\pi}{128}.
\end{aligned}$$

3. Let us denote $u_1 = x^2 + y^2 - z^2 - 2$ and $u_2 = x + y - 2e^z$. It is straightforward to check that $u_1(1,1,0) = u_2(1,1,0) = 0$. Moreover,

$$\left.\frac{\partial(u_1, u_2)}{\partial(y, z)}\right|_{(1,1,0)} = \left.\begin{vmatrix} 2y & -2z \\ 1 & -2e^z \end{vmatrix}\right|_{(1,1,0)} = \begin{vmatrix} 2 & 0 \\ 1 & -2 \end{vmatrix} = -4.$$

Since $-4 \neq 0$ we can appeal to the implicit function theorem to obtain the conclusion that the system can be solved for y and z as C^1-smooth functions of x near $(1,1,0)$. Differentiating through with respect to x in the equations gives $2x + 2y(x)y'(x) - 2z(x)z'(x) = 0$ and $1 + y'(x) - 2e^{z(x)}z'(x) = 0$. Inserting $(x,y,z) = (1,1,0)$ we get $2 + 2y'(1) = 0$ and $1 + y'(1) - 2z'(1) = 0$, so $y'(1) = -1$ and $z'(1) = 0$.

4. Writing $u = u(x,y) = u(s,t)$ for the dependent variable, we have

$$u'_x = u'_s s'_x + u'_t t'_x = \frac{1}{\cos^2 x} u'_s - \frac{1}{\cos^2 x} u'_t$$

and

$$u'_y = u'_s s'_y + u'_t t'_y = u'_t.$$

This gives (assuming C^2-smoothness)

$$u''_{xy} = \frac{1}{\cos^2 x} u''_{st} - \frac{1}{\cos^2 x} u''_{tt}$$

and

$$u''_{yy} = u''_{tt}.$$

It follows that

$$\cos^2 x \cdot u''_{xy} + u''_{yy} = u''_{st},$$

so the differential equation is equivalent to $u''_{st} = 0$, which has the general solution

$$u(x,y) = F(s) + G(t) = F(\tan x) + G(y - \tan x).$$

To solve the initial value problem $u(x,0) = \tan^2 x$, $u'_y(x,0) = 0$ we need $F(\tan x) + G(-\tan x) = \tan^2 x$ and $G'(-\tan x) = 0$. This is satisfied with $F(t) = t^2$ and $G(t) = 0$. An answer is thus

$$u(x,y) = \tan^2 x.$$

5. Let $(x(t), y(t))$ be a parametrization of the given curve. The tangent vector $(x'(t), y'(t))$ must at all points be parallel to the gradient $\nabla f(x(t), y(t)) = (2x(t)e^{y(t)}, x(t)^2 e^{y(t)})$. There is hence a scalar $\lambda(t)$ such that $(x'(t), y'(t)) = \lambda(t)(2x(t)e^{y(t)}, x(t)^2 e^{y(t)})$. Solving for $\lambda(t)$ gives

$$\frac{x'(t)}{2x(t)e^{y(t)}} = \frac{y'(t)}{x(t)^2 e^{y(t)}} \quad \Longleftrightarrow \quad x(t)x'(t) = 2y'(t).$$

The latter equation has general solution $x(t)^2/2 = 2y(t) + C$, i.e., the curve must be of the form $y = x^2/4 + D$ for some constant D. Since it passes through $(1,1)$ we obtain $D = 3/4$, so the curve is $4y = x^2 + 3$.

6. (a) Check the course literature.

(b) Integrating in Taylor's formula, we have as $\epsilon \to 0+$ that

$$\begin{aligned}
A(\epsilon, u) &= \frac{1}{\pi\epsilon^2} \iint_{D_\epsilon} P_2(x, y)\, dxdy + O(\epsilon^3) \\
&= \frac{1}{\pi\epsilon^2} \iint_{D_\epsilon} (u(0,0) + u_1'(0,0)x + u_2'(0,0)y + \\
&\qquad + \frac{1}{2}(u_{11}''(0,0)x^2 + u_{22}''(0,0)y^2 + 2u_{12}''(0,0)xy))\, dxdy + O(\epsilon^3) \\
&= u(0,0) + \frac{1}{2}\frac{1}{\pi\epsilon^2} \iint_{D_\epsilon} (u_{11}''(0,0)x^2 + u_{22}''(0,0)y^2)\, dxdy + O(\epsilon^3),
\end{aligned}$$

where we used symmetry to identify several integrals. We now compute

$$\begin{aligned}
\iint_{D_\epsilon} x^2\, dA &= \iint_{D_\epsilon} y^2\, dA \\
&= \frac{1}{2} \iint_{D_\epsilon} (x^2 + y^2)\, dA \\
&= \pi \int_0^\epsilon r^3\, dr = \pi\epsilon^4/4.
\end{aligned}$$

We have shown that

$$A(\epsilon, u) = u(0,0) + \frac{\epsilon^2}{8}\Delta u(0,0) + O(\epsilon^3), \quad (\epsilon \to 0+),$$

and so $A(\epsilon, u) - u(0,0) = \epsilon^2 \Delta u(0,0)/8 + O(\epsilon^3) > 0$ if $\epsilon > 0$ is small enough.