LUND UNIVERSITY

Centre for Mathematical Sciences
Mathematics, Faculty of Science

Written Examination
Analysis of Several Variables 1
Saturday 29 October 2016
Duration: 14:00–19:00

Solutions

1. We differentiate the function and determine the critical points:

$$\begin{array}{rcr} \partial f/\partial x &=& 2x+4y=0 \\ \partial f/\partial y &=& 4x+48y-5y^4=0 \end{array} \Leftrightarrow \begin{array}{rcr} x &=& -2y \\ 40y-5y^4 &=& 0 \end{array}$$

The second equation has two solutions, $y = 0$ and $y = 2$, so we have two critical points, $(0, 0)$ and $(-4, 2)$. To classify them, we need the second derivatives:

$$\frac{\partial^2 f}{\partial x^2} = 2, \quad \frac{\partial^2 f}{\partial y^2} = 48 - 20y^3, \quad \frac{\partial^2 f}{\partial y \partial x} = 4.$$

For $(0, 0)$, the quadratic form of second derivatives is

$$Q(h, k) = 2h^2 + 8hk + 48k^2 = 2(h + 2k)^2 + 40k^2,$$

which is positive definite, so $(0, 0)$ is a local minimum. For $(-4, 2)$, the quadratic form of second derivatives is

$$Q(h, k) = 2h^2 + 8hk - 112k^2 = 2(h + 2k)^2 - 120k^2,$$

which is indefinite, so $(-4, 2)$ is a saddle.
Answer: The only local extreme point is $(0, 0)$, a local minimum.

2. The inequality $x \geq \sqrt{3}|y|$ is equivalent to

$$-\frac{x}{\sqrt{3}} \leq y \leq \frac{x}{\sqrt{3}}, \quad x \geq 0,$$

so D is the sector of the unit circle where $-\pi/6 \leq \theta \leq \pi/6$. We change variables to polar coordinates and find that the integral is equal to

$$\int_{-\pi/6}^{\pi/6} \cos\theta d\theta \int_0^1 \frac{r^2}{1 + r^2} dr.$$

We have

$$\int_{-\pi/6}^{\pi/6} \cos\theta d\theta = \Big[\sin\theta\Big]_{-\pi/6}^{\pi/6} = (1/2 - (-1/2)) = 1,$$

and

$$\int_0^1 \frac{r^2}{1 + r^2} dr = \int_0^1 1 - \frac{1}{1 + r^2} dr = \Big[r - \arctan r\Big]_0^1 = 1 - \frac{\pi}{4}.$$

Answer: The value of the integral is $1 - \frac{\pi}{4}$.

Please, turn over!

3. The function is of class C^1 and the disk $x^2 + y^2 \leq 9$ is closed and bounded. The maximum and minimum are therefore attained either in an inner critical point, or on the boundary of the disk. We differentiate the function to find the critical points:

$$\frac{\partial f}{\partial x} = \frac{4 - 4x^2 + 4y^2 + 6x}{(1 + x^2 + y^2)^2} = 0, \quad \frac{\partial f}{\partial y} = \frac{-2y(4x - 3)}{(1 + x^2 + y^2)^2} 0$$

The second equation is equivalent to $y = 0$ or $x = 3/4$. Insertion of $y = 0$ into the first equation gives the quadratic equation

$$-4x^2 + 6x + 4 = 0,$$

which has the solutions $x = 2$ and $x = -1/2$. Insertion of $x = 3/4$ gives

$$1 + \frac{9}{16} + y^2 = 0,$$

which has no solutions. Thus there are two critical points, $(2, 0)$ and $(-1/2, 0)$, both inner points of the disk. The values of f at these points are

$$f(2, 0) = 1, \qquad f(-1/2, 0) = -4.$$

On the boundary where $x^2 + y^2 = 9$ we have

$$f(x, y) = \frac{1}{10}(4x - 3).$$

The maximal value of x on the boundary is $x = 3$ and the minimal value is $x = -3$. Therefore, the maximal value of f there is 0.9 which is less than 1 and the minimal value is -1.5 which is greater than -4. The maximum and minimum values of f in the closed disk are therefore attained at the inner critical points.

Answer: The minimum is -4 and the maximum is 1.

4. The point $(2, 1)$ satisfies the equation

$$f(x, y) = x^3 y + x \ln y + y^3 = 9.$$

We have

$$\frac{\partial f}{\partial y} = x^3 + x/y + 3y^2,$$

which is equal to $13 \neq 0$ at $(2, 1)$. By the implicit function theorem, the equation then defines y as a function of x in a neighborhood of $x = 2$ and $y(2) = 1$. The function y is of class C^2 (and even C^∞) since f is. To find $y'(2)$ we differentiate the equation:

$$3x^2 y + x^3 y' + \ln y + \frac{xy'}{y} + 3y^2 y' = 0.$$

This equation can be solved for y':

$$y'(x^3 + \frac{x}{y} + 3y^2) = -3x^2 y - \ln y.$$

By inserting $x = 2$ and $y = 1$ we find that $y'(2) = -12/13$. To find y'', we differentiate the last equation again:

$$y''(x^3 + \frac{x}{y} + 3y^2) + y'(3x^2 + \frac{1}{y} - \frac{xy'}{y^2} + 6yy') = -6xy - 3x^2 y' - \frac{y'}{y}.$$

When we insert $x = 2$, $y = 1$ and $y' = -12/13$, we find that the right hand side is 0 and

$$13y'' + y'(13 + 4y') = 0, \qquad y''(2) = \frac{12}{13} - \frac{4 * 12^2}{13^3} = \frac{12 * 121}{13^3}.$$

Answer: $y'(2) = -\frac{12}{13}$, $y''(2) = \frac{1452}{2197}$.

5. We differentiate the equation and obtain

$$-2f(x) + \int_0^x 2(x-t)f(t)dt = 4.$$

This equation can be differentiated again:

$$-2f'(x) + 2\int_0^x f(t)dt = 0,$$

and one more differentiation gives

$$-2f''(x) + 2f(x) = 0.$$

The general solution of this differential equation is

$$f(x) = C_1 e^x + C_2 e^{-x}.$$

The first equation above shows that $f(0) = -2$ and the second shows that $f'(0) = 0$. This implies that $C_1 = C_2 = -1$.
Answer: $f(x) = -e^x - e^{-x} = -2\cosh x$.

6. We change variables to spherical coordinates:

$$\begin{aligned} x &= r\cos\theta\sin\phi, \\ y &= r\sin\theta\sin\phi, \\ z &= r\cos\phi. \end{aligned}$$

We have $dxdydz = d\phi dR d\theta$ and the unit sphere is described by

$$0 \le R \le 1, \quad 0 \le \theta \le 2\pi, \quad 0 \le \phi \le \pi.$$

The integral is equal to

$$I = \int_0^{2\pi}\int_0^{\pi}\int_0^1 |R^2\sin^2\phi - R^2\cos^2\phi| R^2 \sin\phi dR d\phi d\theta$$

The integration by θ gives a factor 2π, and the integration by R gives a factor $1/5$ so we have

$$I = \frac{2\pi}{5}\int_0^{\pi} |\sin^2\phi - \cos^2\phi|\sin\phi d\phi.$$

Since

$$\sin^2\phi - \cos^2\phi = 1 - 2\cos^2\phi,$$

the substitution $t = \cos\phi$ gives

$$I = \frac{4\pi}{5}\int_{-1}^1 |1 - 2t^2| dt.$$

By symmetry,

$$I = \frac{4pi}{5}\int_0^1 |1-2t^2|dt = \frac{4\pi}{5}\int_0^{1/\sqrt{2}} 1 - 2t^2 dt - \frac{4\pi}{5}\int_{1/\sqrt{2}}^1 1 - 2t^2 dt,$$

which is simple to evaluate. **Answer:** The value of the integral is $\frac{4\pi}{15}(2\sqrt{2} - 1)$.