SOLUTIONS TO THE EXAM 160322

1. We compute $f'_x = 3x^2 + 3y^2 + 4x$ and $f'_y = 6xy - 2y$. The critical points are calculated to $(0,0)$ and $(-4/3, 0)$. The quadratic form at $(0,0)$ is $Q(h,k) = 4h^2 - 2k^2$ which is indefinite, i.e. $(0,0)$ is a saddle-point. At $(-4/3, 0)$ the quadratic form is $Q(h,k) = -8h^2 - 10k^2$ which is negative definite. Hence we have a local maximum at $(-4/3, 0)$.

2. Let $F(x,y,z) = x^3 + y^3 + z^3 - xyz - 26$. Then $F'_z = 3z^2 - xy$ so $F'_z(1,1,3) = 26 \neq 0$. The implicit function theorem now implies that there is a C^1-function $z(x,y)$ defined in a neighbourhood of $(1,1)$ such that

$$x^3 + y^3 + z(x,y)^3 - xyz(x,y) = 26.$$

Differentiating here with respect to x gives $3x^2 + 3z^2 z'_x - yz - xyz'_x = 0$. Inserting $(x,y,z) = (1,1,3)$ gives $3 + 26z'_x(1,1) - 3 = 0$ so $z'_x(1,1) = 0$. Similarly we compute $z'_y(1,1) = 0$. Hence $\nabla z(1,1) = (0,0)$ i.e., $(1,1)$ is a critical point.

3. By symmetry of the domain, the given integral simplifies to

$$I = \iiint_D z^2 \, dxdydz.$$

In space-polar coordinates, the domain D corresponds to $\tilde{D} : r \leq 2, \; \pi/4 \leq \phi \leq 3\pi/4$. Hence

$$\begin{aligned} I &= \iiint_{\tilde{D}} r^2 \cos^2\phi \cdot r^2 \sin\phi \, dr d\theta d\phi \\ &= \int_0^{2\pi} d\theta \cdot \int_0^2 r^4 \, dr \cdot \int_{\pi/4}^{3\pi/4} \cos^2\phi \sin\phi \, d\phi \\ &= 2\pi \cdot \frac{2^5}{5} \cdot \left[-\frac{\cos^3\phi}{3} \right]_{\pi/4}^{3\pi/4} = \frac{32\pi\sqrt{2}}{15}. \end{aligned}$$

4. Write $v(s,t) = u(x,y)$ where $s = \sqrt{y}$ and $t = \sqrt{x-y}$. We compute

$$u'_x = \frac{\partial v}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial v}{\partial t}\frac{\partial t}{\partial x} = v'_t \cdot \frac{1}{2t}$$

and

$$u''_{xx} = (v''_{st}\frac{\partial s}{\partial x} + v''_{tt}\frac{\partial t}{\partial x}) \cdot \frac{1}{2t} + v'_t \cdot (-\frac{1}{2t^2})\frac{\partial t}{\partial x} = v''_{tt} \cdot \frac{1}{4t^2} - v'_t \frac{1}{4t^3}.$$

Finally,

$$u''_{xy} = (v''_{st}\frac{\partial s}{\partial y} + v''_{tt}\frac{\partial t}{\partial y}) \cdot \frac{1}{2t} + v'_t \cdot (-\frac{1}{2t^2}) \cdot \frac{\partial t}{\partial y} = v''_{st} \cdot \frac{1}{4st} - v''_{tt}\frac{1}{4t^2} + \frac{1}{4t^3}v'_t.$$

It follows that

$$u''_{xx} + u''_{xy} = \frac{v''_{st}}{4st}, \quad s, t > 0$$

so in particular $u''_{xx} + u''_{xy} = 0$ corresponds to $v''_{st} = 0$. Integrating the last equation we get $v'_s = \phi(t)$ and then $v = \Phi(t) + \Psi(s)$ where $\Phi' = \phi$. This gives the answer

$$u(x, y) = \Phi(\sqrt{x - y}) + \Psi(\sqrt{y})$$

where Φ and Ψ are arbitrary (C^2-smooth) functions of one variable.

5. Consider the auxiliary function $g(u, v) = \int_1^u \frac{\ln(tv)}{1+t^2}\, dt$. Then $f(x) = g(2x, x)$ so $f'(1) = 2g'_1(2, 1) + g'_2(2, 1)$. By the fundamental theorem of calculus we have

$$g'_1(u, v) = \frac{\ln(uv)}{1 + u^2}$$

so $g'_1(2, 1) = \frac{\ln 2}{5}$. By differentiation under the integral sign,

$$g'_2(u, v) = \frac{1}{v}\int_1^u \frac{dt}{1 + t^2} = \frac{1}{v}(\arctan u - \pi/4).$$

It follows that

$$f'(1) = 2\frac{\ln 2}{5} + \arctan 2 - \frac{\pi}{4}.$$

6. For reasons of symmetry, the arclength is four times the length of the piece of the curve in the first quadrant. This curve is parametrized by $x = \cos^3 t$, $y = \sin^3 t$, $0 \le t \le \pi/2$. The arclength element is

$$\begin{aligned} ds &= \sqrt{x'(t)^2 + y'(t)^2}\, dt = \sqrt{9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t}\, dt \\ &= 3 \sin t \cos t\, dt = \frac{3}{2}\sin 2t\, dt. \end{aligned}$$

It follows that the arclength in question is

$$4\int_0^{\pi/2} \frac{3}{2}\sin 2t\, dt = 4\left[-\frac{3}{4}\cos(2t)\right]_0^{\pi/2} = 6.$$