LUND UNIVERSITY

Written Examination
Linear analysis
Wednesday 16 March, 2017
Duration: 8:00–13:00

Centre for Mathematical Sciences
Mathematics, Faculty of Science

Solutions

1. **a)** The series

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{\ln k}$$

is **convergent**, since it satisfies the conditions of Leibniz' criterion for convergence:

- The terms are of *alternate sign.*
- The *terms tend to zero* as k tends to ∞
- The absolute value of the terms is a *decreasing function* of k because the denominator $\ln k$ is an increasing function of k.

b) The series

$$\sum_{k=1}^{\infty} \frac{2^k k!}{k^k}$$

is **convergent**. With $a_k = \frac{2^k k!}{k^k}$ we have

$$\frac{a_{k+1}}{a_k} = \frac{2k^k}{(k+1)^k} = 2(1+1/k)^{-k} \to \frac{2}{e} \quad \text{as} \quad k \to \infty.$$

The convergence now follows from the ratio test since the limit is less than 1.

c) The series

$$\sum_{k=1}^{\infty} \frac{(2+3i)^k}{(3+2i)^k}$$

is **divergent** since the absolute value of the term is

$$\frac{|2+3i|^k}{|3+2i|^k} = 1$$

so the terms do not tend to zero.

2. Let $v(x,t) = u(x,t) - x$. Then v solves the problem

$$\begin{cases} \partial_t v(x,t) = 3\partial_{xx}^2 v(x,t), & 0 < x < \pi, \quad t > 0, \\ \partial_x v(0,t) = \partial_x v(\pi,t) = 0, & t > 0, \\ v(x,0) = \cos 4x \cos 2x, & 0 < x < \pi. \end{cases}$$

By Euler's formulas,

$$\cos 4x \cos 2x = \Big(\frac{e^{4ix}+e^{-4ix}}{2}\Big)^2\Big(\frac{e^{2ix}+e^{-2ix}}{2}\Big) = \frac{\cos 2x}{2} + \frac{\cos 6x}{2}.$$

Please, turn over!

Hence the solution is

$$v(x,t) = \frac{1}{2}(e^{-12t}\cos 2x + e^{-108t}\cos 6x).$$

and

$$u(x,t) = x + \frac{1}{2}(e^{-12t}\cos 2x + e^{-108t}\cos 6x).$$

3. a) By using Euler's formula for $\sin x$, we find that

$$c_n = \frac{1}{4\pi i}\int_{-\pi}^{\pi} e^{x(1+i-in)} - e^{x(1-i-in)}dx = \frac{1}{4\pi i}\Big[\frac{e^{x(1+i-in)}}{1+i(1-n)} - \frac{e^{x(1-i-in)}}{1-i(1+n)}\Big]_{-\pi}^{\pi}$$

Inserting π and $-\pi$, and using that $e^{ik\pi} = (-1)^k$ for any integer k and $\sinh x = (e^x - e^{-x})/2$ gives

$$c_n = \frac{(-1)^{n+1}\sinh\pi}{2\pi i}\Big(\frac{1}{1+i(1-n)} - \frac{1}{1-i(1+n)}\Big) = \frac{(-1)^{n+1}\sinh\pi}{\pi}\frac{1}{n^2+2in-2}$$

Thus the Fourier series of u is

$$u(x) = \frac{\sinh\pi}{\pi}\sum_{k=-\infty}^{\infty}\frac{(-1)^{n+1}e^{inx}}{n^2+2in-2}$$

The function u is equal to the sum of its Fourier series, since it is continuous and piecewise C^1.

b) The sum of the series for $x = 3\pi/2$ is $u(3\pi/2)$. Since u has period 2π,

$$u(3\pi/2) = u(-\pi/2) = e^{-\pi/2}\sin(-\pi/2) = -e^{-\pi/2}.$$

c) First note that

$$\frac{1}{n^2+2in-2} = \frac{n^2-2in-2}{(n^2-2)^2+4n^2} = \frac{n^2-2}{n^4+4} + i\frac{-2n}{n^4+4}$$

The terms of the series in question are thus the real parts of the Fourier coefficients. For $x = \pi$, we have

$$u(\pi) = 0 = \frac{-\sinh\pi}{\pi}\sum_{k=-\infty}^{\infty}\frac{n^2-2}{n^4+4} + i\frac{-2n}{n^4+4}$$

The sum of the real parts is zero, so

$$0 = -\frac{1}{2} + 2\sum_{k=1}^{\infty}\frac{n^2-2}{n^4+4}.$$

It follows that

$$\sum_{n=1}^{\infty}\frac{n^2-2}{n^4+4} = \frac{1}{4}$$

4. Assume that u is given by a power series with a positive radius of convergence, $u(x) = \sum_{k=0}^{\infty} a_k x^k$. We can differentiate term by term if x is within the radius of convergence:

$$u'(x) = \sum_{k=1}^{\infty} k a_k x^{k-1}, \qquad u''(x) = \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2}$$

After insertion in the differential equation we get

$$\sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} - \sum_{k=1}^{\infty} k a_k x^k + \sum_{k=0}^{\infty} a_k x^k = 0.$$

We replace $k-2$ by k in the first series:

$$\sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k + \sum_{k=1}^{\infty} k a_k x^k + \sum_{k=0}^{\infty} a_k x^k = 0.$$

If the equality is valid for all x in a neighborhood of 0 then the coefficient for every power of x is zero. It follows that

$$(k+2) a_{k+2} + a_k = 0, \quad k \geq 0.$$

The initial values imply that $a_0 = 1$ and $a_1 = 0$. Hence all coefficients with odd indices are zero and

$$a_{2k} = \frac{(-1)^k}{2^k k!}.$$

The solution is

$$u(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^k k!} = e^{-x^2/2}.$$

5. **a)** For $x \geq 0$ we have

$$0 \leq \frac{x}{2 + k^3 x} \leq \frac{1}{k^3}$$

so the series is uniformly convergent for $x \geq 0$ by Weierstrass' M-test since $\sum k^{-3}$ is convergent. It follows that f is continuous for $x \geq 0$.

b) The derivative of term number k is

$$g_k(x) = \frac{2}{(2 + k^3 x)^2}$$

Let $a > 0$. The supremum of $g_k(x)$ for $x \geq a$ is less than or equal to $\frac{2}{(2+ak^3)^2}$ and the series

$$\sum_{k=1}^{\infty} \frac{2}{(2 + ak^3)^2}$$

is convergent since $a \neq 0$. By Weierstrass' M-test, the differentiated series is uniformly convergent for $x \geq a$. Therefore, the function f is differentiable for $x \geq a$. Since this is true for any $a > 0$, it follows that f is differentiable for $x > 0$.

c) We have $f(0) = 0$. If f has a right derivative at 0, it is equal to

$$\lim_{x \to 0} \frac{f(x)}{x} = \lim_{x \to 0} \sum_{k=1}^{\infty} \frac{1}{(2 + k^3 x)^2}$$

Let N be any positive integer. Since the terms of the series are positive, the limit is greater than the limit of the first N terms, which is equal to $N/4$. Since this is true for any N, the limit is ∞ and the derivative does not exist.