LUND UNIVERSITY

Written Examination
Linear analysis
Saturday 28 May 2016
Duration: 8:00–13:00

Centre for Mathematical Sciences
Mathematics, Faculty of Science

Solutions

1. a) The series $\sum_{k=1}^{\infty} k\tan(\frac{1}{k^3})$ is **convergent** since

$$\frac{k\tan(\frac{1}{k^3})}{\frac{1}{k^2}} \to 1 \quad \text{as} \quad k\to\infty$$

and $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is convergent.

b) The series $\sum_{k=1}^{\infty} \frac{(k!)^2}{(2k)!}$ is **convergent**. With $a_k = \frac{(k!)^2}{(2k)!}$ we have

$$\frac{a_{k+1}}{a_k} = \frac{(k+1)^2}{(2k+2)(2k+1)} \to \frac{1}{4} \quad \text{as} \quad k\to\infty.$$

The convergence now follows from the ratio test since the limit is less than 1.

c) The series $\sum_{k=1}^{\infty} \frac{(-1)^k}{\ln k+(-1)^k/k^2}$ is **convergent**, since it satisfies the conditions of Leibniz' criterion for convergence:

- The terms are of *alternate sign* from term number two and onwards.
- The *terms tend to zero* as k tends to ∞ since $\ln k$ tends to ∞.
- We need to prove that the absolute value of the terms is a *decreasing function* of k, or equivalently that the denominator $g(k) = \ln k + (-1)^k/k^2$ is *increasing.* By the mean value theorem, $\ln(k+1) - \ln k \geq 1/(k+1)$. Therefore,

$$g(k+1) - g(k) \geq \frac{1}{k+1} + \frac{(-1)^{k+1}}{(k+1)^2} - \frac{(-1)^k}{k^2},$$

which is positive for large enough k.

2. By Euler's formulas,

$$(\cos x)^4 = \Big(\frac{e^{ix}+e^{-ix}}{2}\Big)^4 = \frac{3}{8} + \frac{\cos 2x}{2} + \frac{\cos 4x}{8}.$$

Hence the solution is

$$\frac{3}{8} + \frac{1}{2}e^{-12t}\cos 2x + \frac{1}{8}e^{-48t}\cos 4x.$$

3. a) By using Euler's formula for $\sin x/2$, we find that the Fourier coefficient c_n is

$$\frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}dx = \frac{1}{2\pi}\int_0^{2\pi} \sin(x/2)e^{-inx}dx = \frac{1}{4\pi i}\int_0^{2\pi} e^{ix(1/2-n)} - e^{-ix(1/2+n)}dx.$$

Please, turn over!

The exponential functions can easily be integrated and the result is

$$c_n = \frac{1}{2\pi}\frac{1}{1/4-n^2}$$

for any n. It follows that the Fourier series is

$$f(x) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} e^{inx}\frac{1}{1/4-n^2}$$

The function f is equal to the sum of its Fourier series, since it is continuous and piecewise C^1.

b) The value of the series for $x=-\pi$ is $f(-\pi)=f(\pi)=\sin(\pi/2)=1$.

c) We have

$$f(0) = 0 = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\frac{1}{1/4-n^2} = \frac{2}{\pi}+\frac{2}{2\pi}\sum_{n=1}^{\infty}\frac{1}{1/4-n^2}.$$

It follows that

$$\sum_{n=1}^{\infty}\frac{1}{n^2-1/4} = 2.$$

d) By Parseval's formula, we have

$$\frac{1}{2\pi}\int_0^{2\pi}\sin^2(x/2)dx = \frac{1}{4\pi^2}\sum_{n=-\infty}^{\infty}\frac{1}{(1/4-n^2)^2}$$

The value of the integral is π, so

$$\frac{1}{2} = \frac{1}{4\pi^2}(2\sum_{n=1}^{\infty}\frac{1}{(1/4-n^2)^2}+16),$$

and it follows that

$$\sum_{n=1}^{\infty}\frac{1}{(1/4-n^2)^2} = \pi^2-8.$$

4. Assume that u is given by a power series with a positive radius of convergence, $u(x) = \sum_{k=0}^{\infty} a_k x^k$. We can differentiate term by term if x is within the radius of convergence:

$$u'(x) = \sum_{k=1}^{\infty} k a_k x^{k-1}, \qquad u''(x) = \sum_{k=2}^{\infty} k(k-1)a_k x^{k-2}$$

After insertion in the differential equation we get

$$2\sum_{k=2}^{\infty} k(k-1)a_k x^{k-1} + \sum_{k=1}^{\infty} k a_k x^{k-1} - 2\sum_{k=1}^{\infty} k a_k x^k + \sum_{k=0}^{\infty} a_k x^k = 0.$$

We replace $k-1$ by k in the first two series and get

$$2\sum_{k=1}^{\infty}(k+1)k a_{k+1}x^k + \sum_{k=0}^{\infty}(k+1)a_{k+1}x^k - 2\sum_{k=1}^{\infty} k a_k x^k + \sum_{k=0}^{\infty} a_k x^k = 0.$$

If the equality is valid for all x in a neighborhood of 0 then the coefficient for every power of x is zero. The coefficient for the power zero is $a_1 + a_0 = 0$. The initial value $u(0) = 1$ implies that $a_0 = 1$, so a_1 is -1. For $k \geq 1$, the coefficient is

$$(2(k+1)k + (k+1))a_{k+1} + (1-2k)a_k = 0,$$

and we get the recursion formula

$$a_{k+1} = \frac{2k-1}{(2k+1)(k+1)} a_k$$

By computing a few of the first a_k, we arrive at the formula

$$a_k = \frac{-1}{k!(2k-1)},$$

which can be proved by induction. The radius of convergence for the power series is ∞, since $a_{k+1}/a_k \to 0$ as $k \to \infty$.

5. **a)** Let $a > 0$. For any $x \geq a$, we have

$$0 \leq \frac{k^2 x}{1 + k^4 x^2} \leq \frac{1}{a k^2}.$$

By Weierstrass M-test, the series is uniformly convergent, and hence continuous on the interval $x \geq a$. Since this is true for any $a > 0$, the function is continuous for $x > 0$.

b) Obviously, $f(0) = 0$. Let n be a positive integer. Since all the terms are positive, the inequality

$$f(x) > \frac{n^2 x}{1 + n^4 x^2}$$

holds for any $x > 0$. In particular, for $x = 1/n^2$, we have

$$f(\frac{1}{n^2}) > \frac{n^2 x}{1 + n^4 x^2} = \frac{1}{2}.$$

Therefore, the limit of $f(x)$ as $x \to 0$ is not $f(0) = 0$, and thus f is **not continuous** at 0.