Written Examination
Linear analysis
Thursday 7 January 2016
Duration: 8:00–13:00

LUND UNIVERSITY

Centre for Mathematical Sciences
Mathematics, Faculty of Science

Solutions

1. **a)** The series $\sum_{k=1}^{\infty} \tan(\frac{1}{k})$ is **divergent** since

$$\tan(\frac{1}{k}) \geq \frac{1}{k}$$

and $\sum_{k=1}^{\infty} \frac{1}{k}$ is divergent.

b) The series $\sum_{k=1}^{\infty} \frac{k^5}{2^k}$ is **convergent**. With $a_k = \frac{k^5}{2^k}$ we have

$$\frac{a_{k+1}}{a_k} = \frac{(k+1)^5}{2k^5} \to \frac{1}{2} \quad \text{as} \quad k \to \infty.$$

The convergence now follows from the ratio test since the limit is less than 1.

c) The series $\sum_{k=1}^{\infty} \frac{(-1)^k}{\ln k + \frac{1}{k}}$ is **convergent**, since it satisfies the conditions of Leibniz' criterion for convergence:

- The terms are of *alternate sign.*
- The *terms tend to zero* as k tends to ∞
- The absolute value of the terms is a *decreasing function* of k because the denominator $\ln k + \frac{1}{k}$ has a positive derivative and therefore is an increasing function of k.

2. **a)** The Fourier coefficient c_n is

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} u(x)e^{-inx}dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} |x|(\cos nx + i \sin nx)dx = \frac{1}{\pi}\int_{0}^{\pi} x \cos nx dx.$$

It follows that $c_0 = \frac{\pi}{2}$. For $n \neq 0$, an integration by parts gives

$$c_n = \frac{-2}{\pi n^2} \quad \text{for} \quad \text{n} \quad \text{odd,} \qquad c_n = 0 \quad \text{for} \quad \text{n} \quad \text{even.}$$

Thus the Fourier series of u is

$$u(x) = \frac{\pi}{2} - \frac{2}{\pi}\sum_{k=-\infty}^{\infty} \frac{e^{i(2k+1)x}}{(2k+1)^2} = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{\cos(2k+1)x}{(2k+1)^2}$$

The function u is equal to the sum of its Fourier series, since it is continuous and piecewise C^1.

Please, turn over!

b) For $x = 0$, we have

$$u(0) = 0 = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}.$$

It follows that

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}.$$

3. By Euler's formulas,

$$(\sin x)^2 \cos 2x = \Big(\frac{e^{ix} - e^{-ix}}{2i}\Big)^2 \Big(\frac{e^{2ix} + e^{-2ix}}{2}\Big) = -\frac{1}{4} + \frac{\cos 2x}{2} - \frac{\cos 4x}{4}.$$

Hence the solution is

$$\frac{1}{4}(-1 + 2e^{-20t}\cos 2x - e^{-80t}\cos 4x).$$

4. Assume that u is given by a power series with a positive radius of convergence, $u(x) = \sum_{k=0}^{\infty} a_k x^k$. We can differentiate term by term if x is within the radius of convergence:

$$u'(x) = \sum_{k=1}^{\infty} k a_k x^{k-1}, \qquad u''(x) = \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2}$$

After insertion in the differential equation we get

$$\sum_{k=2}^{\infty} k(k-1)a_k x^{k-2} + \sum_{k=1}^{\infty} k a_k x^k + \sum_{k=0}^{\infty} a_k x^k = 0.$$

We replace $k-2$ by k in the first series:

$$\sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2} x^k + \sum_{k=1}^{\infty} k a_k x^k + \sum_{k=0}^{\infty} a_k x^k = 0.$$

If the equality is valid for all x in a neighborhood of 0 then the coefficient for every power of x is zero. It follows that

$$(k+2)a_{k+2} + a_k = 0, \quad k \geq 0.$$

The initial values imply that $a_0 = 1$ and $a_1 = 0$. Hence all coefficients with odd indices are zero and

$$a_{2k} = \frac{(-1)^k}{2^k k!}.$$

The solution is

$$u(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^k k!} = e^{-x^2/2}.$$

5. **a)** If $x = 0$, all terms are zero. If $x \neq 0$ we have

$$0 \leq \frac{|\sin kx|}{1 + k^2x^2} \leq \frac{1}{1 + k^2x^2} \leq \frac{1}{k^2x^2},$$

so the series is convergent since $\sum k^{-2}$ is convergent.

b) Let $a > 0$. The supremum of $\frac{|\sin kx|}{1+k^2x^2}$ for $|x| \geq a$ is less than or equal to $\frac{1}{1+a^2k^2}$ and the series

$$\sum_{k=1}^{\infty} \frac{1}{1+a^2k^2}$$

is convergent since $a \neq 0$. By Weierstrass' M-test, the series is uniformly convergent, and therefore continuous for $|x| \geq a$. Since this is true for any $a \neq 0$, it follows that s is continuous for $x \neq 0$.

c) The function s is not continuous at 0. We have $s(0) = 0$. Let N be a positive integer. Since the terms of s are positive, $s(x)$ is for any x greater than the term number N. In particular, this is true for $x = 1/N$, so

$$s(1/N) \geq \frac{|\sin N/N|}{1+N^2N^{-2}} = \frac{\sin 1}{1+1}.$$

Therefore, $s(x)$ does not tend to $s(0)$ as x tends to 0.