LUND UNIVERSITY

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Solutions

1. The coefficients of the sine series of $x(\pi - x)$ are given by

$$\begin{aligned}
b_n &= \frac{2}{\pi}\int_0^\pi x(\pi-x)\sin(nx)\,dx \\
&= \frac{2}{\pi}\left(\left[x(\pi-x)\cdot\frac{-\cos(nx)}{n}\right]_0^\pi - \int_0^\pi(\pi-2x)\cdot\frac{-\cos(nx)}{n}\,dx\right) \\
&= \frac{2}{\pi}\int_0^\pi(\pi-2x)\cdot\frac{\cos(nx)}{n}\,dx \\
&= \frac{2}{\pi}\left(\left[(\pi-2x)\cdot\frac{\sin(nx)}{n^2}\right]_0^\pi - \int_0^\pi(-2)\cdot\frac{\sin(nx)}{n^2}\,dx\right) \\
&= \frac{4}{\pi}\int_0^\pi\frac{\sin(nx)}{n^2}\,dx = \frac{4}{\pi}\left[\frac{-\cos(nx)}{n^3}\right]_0^\pi = \frac{4(1-(-1)^n)}{\pi n^3} \\
&= \begin{cases} 0, & \text{if } n \text{ is even,} \\ \dfrac{8}{\pi(2k+1)^3}, & \text{if } n = 2k+1 \text{ is odd.}\end{cases}
\end{aligned}$$

Hence, the solution is

$$u(x,t) = \frac{8}{\pi}\sum_{k=0}^\infty \frac{e^{-t(2k+1)^2}}{(2k+1)^3}\cdot\sin((2k+1)x).$$

2. a) We have that

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^\pi u(x)\,dx = 0$$

since u is an odd function. By using that fact also when $n \neq 0$, we get

$$\begin{aligned}
c_n &= \frac{1}{2\pi}\int_{-\pi}^\pi u(x)(\cos(nx) - i\sin(nx))\,dx = -\frac{i}{\pi}\int_0^\pi(x^3-\pi^2x)\sin(nx)\,dx \\
&= \frac{i}{\pi}\left(\left[(x^3-\pi^2x)\cdot\frac{\cos(nx)}{n}\right]_0^\pi - \int_0^\pi(3x^2-\pi^2)\cdot\frac{\cos(nx)}{n}\,dx\right) \\
&= -\frac{i}{\pi}\left(\left[(3x^2-\pi^2)\cdot\frac{\sin(nx)}{n^2}\right]_0^\pi - \int_0^\pi 6x\cdot\frac{\sin(nx)}{n^2}\,dx\right) \\
&= -\frac{i}{\pi}\left(\left[6x\cdot\frac{\cos(nx)}{n^3}\right]_0^\pi - 6\int_0^\pi\frac{\cos(nx)}{n^3}\,dx\right) = -\frac{i}{\pi}\left(6\pi\cdot\frac{(-1)^n}{n^3}\right) \\
&= -\frac{6i(-1)^n}{n^3}.
\end{aligned}$$

Therefore, the Fourier series is

$$-6i \sum_{n \neq 0} \frac{(-1)^n}{n^3} e^{inx}.$$

b) Parseval's formula now gives that

$$\begin{aligned} 36 \sum_{n \neq 0} \frac{1}{n^6} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (x^3 - \pi^2 x)^2 \, dx = \frac{1}{\pi} \int_0^{\pi} (x^6 - 2\pi^2 x^4 + \pi^4 x^2) \, dx \\ &= \frac{1}{\pi} \left[\frac{x^7}{7} - \frac{2\pi^2 x^5}{5} + \frac{\pi^4 x^3}{3} \right]_0^{\pi} = \frac{8\pi^6}{105} \end{aligned}$$

and hence

$$\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{8\pi^6}{72 \cdot 105} = \frac{\pi^6}{945}.$$

3. The radius of convergence of the power series $\sum_{n=0}^{\infty} x^n$ is 1. Hence we may compute the derivatives of all orders of the function

$$s(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

by means of termwise differentiation when $|x| < 1$. We get

$$s'(x) = \sum_{n=1}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2} \quad \text{and} \quad s''(x) = \sum_{n=2}^{\infty} n(n-1) x^{n-2} = \frac{2}{(1-x)^3}.$$

Hence

$$\sum_{n=1}^{\infty} n x^n = \frac{x}{(1-x)^2} \quad \text{and} \quad \sum_{n=1}^{\infty} n(n-1) x^n = \frac{2x^2}{(1-x)^3}$$

and therefore

$$\sum_{n=1}^{\infty} (n^2 + 2n) x^n = \frac{2x^2}{(1-x)^3} + 3 \cdot \frac{x}{(1-x)^2} = \frac{x(3-x)}{(1-x)^3}.$$

When $x = \frac{1}{3}$ this yields

$$\sum_{n=1}^{\infty} \frac{n^2 + 2n}{3^n} = 3.$$

4. Assume that $y(x) = \sum_{k=0}^{\infty} a_k x^k$ is a power series solution with radius of convergence $R > 0$. When $|x| < R$ we may use termwise differentiation and hence

$$y'(x) = \sum_{k=1}^{\infty} k a_k x^{k-1}, \quad y''(x) = \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2}.$$

After insertion in the differential equation we get

$$\sum_{k=2}^{\infty} 2k(k-1) a_k x^{k-1} + \sum_{k=1}^{\infty} k a_k x^{k-1} - \sum_{k=1}^{\infty} 2k a_k x^k + \sum_{k=0}^{\infty} a_k x^k = 0.$$

We replace k with $k-1$ in the last two series and get

$$\sum_{k=1}^{\infty} 2k(k-1)a_k x^{k-1} + \sum_{k=1}^{\infty} k a_k x^{k-1} - \sum_{k=1}^{\infty} 2(k-1)a_{k-1}x^{k-1} + \sum_{k=1}^{\infty} a_{k-1}x^{k-1} = 0$$

and this is equivalent to

$$\sum_{k=1}^{\infty} (k(2k-1)a_k - (2k-3)a_{k-1})x^{k-1} = 0.$$

Since this equality holds for all small values of x and $y(0) = 1$, we get the recurrence equation

$$a_0 = 1, \qquad a_k = \frac{2k-3}{k(2k-1)}a_{k-1}, \quad k \geq 1.$$

Hence

$$a_k = -\frac{1}{k!(2k-1)}, \quad k \geq 0.$$

The recurrence equation gives that $|\frac{a_{k+1}}{a_k}| \to 0$ as $k \to \infty$. Hence the radius of convergence of this power series is ∞. Consequently,

$$y(x) = -\sum_{k=0}^{\infty} \frac{x^k}{k!(2k-1)}, \quad x \in \mathbf{R},$$

is a solution and, in fact, the only power series solution.

5. The pointwise limit of the sequence is

$$f(x) = \begin{cases} 0, & \text{when } x \in [0,1) \text{ or } x \in (1,\infty), \\ \frac{1}{2}, & \text{when } x = 1. \end{cases}$$

Since the functions f_n are continuous and f is discontinuous at $x = 1$, the convergence cannot be uniform in the interval $[\frac{1}{2}, 2]$. Since

$$f_n(x) \leq \frac{x^n}{1} = x^n \leq \frac{1}{2^n} \quad \text{when} \quad 0 \leq x \leq \frac{1}{2} \qquad \Rightarrow \qquad \|f_n\|_{[0,\frac{1}{2}]} \leq \frac{1}{2^n}$$

and

$$f_n(x) \leq \frac{x^n}{x^{2n}} = \frac{1}{x^n} \leq \frac{1}{2^n} \quad \text{when} \quad x \geq 2 \qquad \Rightarrow \qquad \|f_n\|_{[2,\infty)} \leq \frac{1}{2^n},$$

the convergence is uniform in the other two intervals.