LUND UNIVERSITY

Centre for Mathematical Sciences
Mathematics, Faculty of Science

Linear Analysis
Thursday, 8 January 2015
Duration: 8.00–13.00

Solutions

1. **a)** Since $e^{-1/k^4} \to 1 \neq 0$ as $k \to \infty$, the series is divergent.

b) We have that $a_k = e^{1/k^2} - 1 > 0$, $b_k = 1/k^2 > 0$ and

$$\frac{a_k}{b_k} = \frac{e^{1/k^2} - 1}{1/k^2} \to 1 > 0 \quad \text{as} \quad k \to \infty.$$

Since $\sum_{k=1}^{\infty} b_k$ is convergent, the series $\sum_{k=1}^{\infty} a_k$ is convergent by the comparison theorem.

c) With $a_k = ke^{-k}$ we have that $a_k > 0$ and

$$\frac{a_{k+1}}{a_k} = \frac{(k+1)e^{-k-1}}{ke^{-k}} = \left(1 + \frac{1}{k}\right) e^{-1} \to e^{-1} < 1 \quad \text{as} \quad k \to \infty.$$

The series is therefore convergent by the ratio test.

2. Set $y(x) = \sum_{k=0}^{\infty} a_k x^k$ and assume that the radius of convergence of this power series is $R > 0$. Then we have that $y'(x) = \sum_{k=1}^{\infty} ka_k x^{k-1}$ and $y''(x) = \sum_{k=2}^{\infty} k(k-1)a_k x^{k-2}$ when $|x| < R$. Inserting this in the equation, we get

$$2\sum_{k=2}^{\infty} k(k-1)a_k x^{k-1} + \sum_{k=1}^{\infty} ka_k x^{k-1} - 2\sum_{k=1}^{\infty} ka_k x^k - 2\sum_{k=0}^{\infty} a_k x^k = 0, \quad |x| < R.$$

Replacing k by $k+1$ in the first two series, we may start the summation from $k = 0$ in all four series. Thus, we get the equivalent equation

$$2\sum_{k=0}^{\infty} (k+1)ka_{k+1}x^k + \sum_{k=0}^{\infty} (k+1)a_{k+1}x^k - 2\sum_{k=0}^{\infty} ka_k x^k - 2\sum_{k=0}^{\infty} a_k x^k = 0, \quad |x| < R,$$

which can be written as

$$\sum_{k=0}^{\infty} ((k+1)(2k+1)a_{k+1} - 2(k+1)a_k)x^k = 0, \quad |x| < R.$$

By the uniqueness theorem, the coefficients must be zero. From this and the fact that $y(0) = 1$ it follows that

$$a_0 = 1, \qquad a_{k+1} = \frac{2a_k}{2k+1}, \quad k \geq 0.$$

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For this power series we have that

$$\left|\frac{a_{k+1}}{a_k}\right| = \frac{2}{2k+1} \to 0 \quad \text{as} \quad k \to \infty.$$

Hence its radius of convergence is ∞ and our computations were valid. We also see that

$$a_k = \frac{2^k}{(2k-1)!!}$$

and therefore the solution is

$$y(x) = \sum_{k=0}^{\infty} \frac{2^k x^k}{(2k-1)!!}.$$

3. We have that

$$\begin{aligned}\sin x \cos 3x &= \frac{(e^{ix} - e^{-ix})(e^{3ix} + e^{-3ix})}{2 \cdot 2i} = \frac{e^{4ix} + e^{-2ix} - e^{2ix} - e^{-4ix}}{2 \cdot 2i} \\ &= -\frac{1}{2}\sin 2x + \frac{1}{2}\sin 4x.\end{aligned}$$

By the uniqueness theorem, this is the sine series expansion $\sum_{n=1}^{\infty} b_n \sin nx$ of $\sin x \cos 3x$. Hence the solution is

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-4n^2 t} \sin nx = -\frac{1}{2} e^{-16t} \sin 2x + \frac{1}{2} e^{-64t} \sin 4x.$$

4. The function u is given by

$$u(x) = \begin{cases} \frac{x}{a\pi} & \text{when } 0 \le x < a\pi, \\ \frac{\pi - x}{\pi - a\pi} & \text{when } a\pi \le x \le \pi. \end{cases}$$

Hence the coefficient b_n is given by

$$\frac{\pi b_n}{2} = \int_0^{a\pi} \frac{x}{a\pi} \sin nx \, dx + \int_{a\pi}^{\pi} \frac{\pi - x}{\pi - a\pi} \sin nx \, dx.$$

We have

$$\begin{aligned}\int_0^{a\pi} \frac{x}{a\pi} \sin nx \, dx &= \left[-\frac{x \cos nx}{na\pi}\right]_0^{a\pi} + \int_0^{a\pi} \frac{\cos nx}{na\pi} \, dx \\ &= -\frac{\cos na\pi}{n} + \left[\frac{\sin nx}{n^2 a\pi}\right]_0^{a\pi} = -\frac{\cos na\pi}{n} + \frac{\sin na\pi}{n^2 a\pi}\end{aligned}$$

and

$$\begin{aligned}\int_{a\pi}^{\pi} \frac{\pi - x}{\pi - a\pi} \sin nx \, dx &= \left[-\frac{(\pi - x)\cos nx}{n(\pi - a\pi)}\right]_{a\pi}^{\pi} - \int_{a\pi}^{\pi} \frac{\cos nx}{n(\pi - a\pi)} \, dx \\ &= \frac{\cos na\pi}{n} - \left[\frac{\sin nx}{n^2(\pi - a\pi)}\right]_{a\pi}^{\pi} = \frac{\cos na\pi}{n} + \frac{\sin na\pi}{n^2(\pi - a\pi)}\end{aligned}$$

and therefore

$$b_n = \frac{2}{\pi}\left(\frac{\sin na\pi}{n^2 a\pi} + \frac{\sin na\pi}{n^2(\pi - a\pi)}\right) = \frac{2\sin na\pi}{\pi^2 n^2 (a - a^2)}.$$

Hence the sine series is

$$\frac{2}{\pi^2(a-a^2)}\sum_{n=1}^{\infty}\frac{\sin na\pi}{n^2}\sin nx.$$

We also have

$$\begin{aligned}\frac{2}{\pi}\int_0^{\pi}|u(x)|^2\,dx &= \frac{2}{\pi}\left(\int_0^{a\pi}\frac{x^2}{a^2\pi^2}\,dx+\int_{a\pi}^{\pi}\frac{(\pi-x)^2}{(\pi-a\pi)^2}\,dx\right)\\ &= \frac{2}{\pi}\left(\left[\frac{x^3}{3a^2\pi^2}\right]_0^{a\pi}-\left[\frac{(\pi-x)^3}{3(\pi-a\pi)^2}\right]_{a\pi}^{\pi}\right)=\frac{2}{3}.\end{aligned}$$

It therefore follows from Parseval's formula that

$$\sum_{n=1}^{\infty}\frac{\sin^2 na\pi}{n^4}=\frac{2}{3}\cdot\frac{\pi^4(a-a^2)^2}{4}=\frac{\pi^4a^2(1-a)^2}{6}.$$

5. **a)** Let x be a fixed real number in the interval $[0,1]$. If $f_n(x)\to A$ as $n\to\infty$, then we must have $A=2A-A^2$ and therefore $A=0$ or $A=1$. Completing the square, we can write the recurrence equation as $f_n(x)=1-(1-f_{n-1}(x))^2$. Since $0\le f_0(x)\le 1$, it now follows by induction that $0\le f_n(x)\le 1$ for all natural numbers n. From $f_n(x)-f_{n-1}(x)=f_{n-1}(x)(1-f_{n-1}(x))$ it then follows that the sequence $(f_n(x))_{n=0}^{\infty}$ is increasing. Hence $(f_n(x))_{n=0}^{\infty}$ is bounded and increasing and has therefore a limit as $n\to\infty$. When $x=0$ it is clear that the limit is 0. When $0<x\le 1$, $f_0(x)>0$, and since $f_n(x)$ is increasing, the limit must be 1. Hence $(f_n)_{n=0}^{\infty}$ is pointwise convergent to the function f defined by

$$f(x)=\begin{cases}0 & \text{when } x=0,\\ 1 & \text{when } 0<x\le 1.\end{cases}$$

b) Since the functions f_n are continuous and f is discontinuous in the interval $[0,\frac{1}{2}]$, the convergence cannot be uniform there.

c) We have that $f_0(x)=x\ge\frac{1}{2}=f_0(\frac{1}{2})$ when $\frac{1}{2}\le x\le 1$. If $f_{n-1}(x)\ge f_{n-1}(\frac{1}{2})$, it follows from $f_n(x)=1-(1-f_{n-1}(x))^2$ that $f_n(x)\ge f_n(\frac{1}{2})$. Hence, by induction, $f_n(x)\ge f_n(\frac{1}{2})$ when $\frac{1}{2}\le x\le 1$ for all natural numbers n. Since also $f_n(x)\le 1$ when $\frac{1}{2}\le x\le 1$, we get

$$|f_n(x)-f(x)|=1-f_n(x)\le 1-f_n(\tfrac{1}{2}),\quad \tfrac{1}{2}\le x\le 1$$

and therefore

$$\|f_n-f\|_{[\frac{1}{2},1]}\le 1-f_n(\tfrac{1}{2}).$$

Since $1-f_n(\frac{1}{2})\to 0$ as $n\to\infty$, it follows that the convergence is uniform in $[\frac{1}{2},1]$.