LUND UNIVERSITY

Centre for Mathematical Sciences
Mathematics, Faculty of Science

Written Examination
Linear Analysis
Thursday, 15 January 2015
Duration: 8:00–13:00

In order to sit for the examination you must be enrolled in the course. No aids except the formula sheet provided in the examination hall. Use the papers provided by the department and write on one side of each sheet only. Fill in the cover completely and write your initials on each sheet. Write legibly. Give concise and short arguments.

1. Which of the following series are convergent?

a) $\displaystyle\sum_{k=1}^{\infty} \frac{\sqrt{k}+1}{2k^2-1}$, **b)** $\displaystyle\sum_{k=1}^{\infty} \frac{\sqrt{k}+1}{2k-1}$, **c)** $\displaystyle\sum_{k=1}^{\infty} (-1)^k \frac{\sqrt{k}+1}{2k-1}$.

2. Find a power series solution of the problem

$$xy''(x) + (1-2x)y'(x) - y(x) = 0, \quad y(0) = 1.$$

3. Solve the heat conduction problem

$$\begin{aligned} \partial_t u(x,t) &= 4\partial_x^2 u(x,t), && 0 \le x \le \pi, \quad t > 0, \\ \partial_x u(0,t) &= \partial_x u(\pi,t) = 0, && t > 0, \\ u(x,0) &= \cos x \cos 3x, && 0 \le x \le \pi. \end{aligned}$$

4. Show that the sequence $(f_n)_{n=1}^{\infty}$ is uniformly convergent in the interval $[0,\infty)$ where $f_n(x) = x^2 e^{-nx}$, $x \ge 0$.

5. Let a be a real number in the open interval $(0,1)$, and let u be the 2π-periodic function for which $u(x) = \cos ax$ when $-\pi \le x \le \pi$.

a) Find the Fourier series expansion of u.

b) Prove that

$$\sum_{n=1}^{\infty} \frac{2a}{a^2-n^2} = \pi \cot a\pi - \frac{1}{a}, \quad 0 < a < 1.$$

c) Integrate both sides with respect to a, and conclude that

$$\sum_{n=1}^{\infty} \ln\left(1 - \frac{a^2}{n^2}\right) = \ln\left(\frac{\sin a\pi}{a\pi}\right), \quad 0 < a < 1,$$

providing proper justification.

d) Deduce the identity

$$\frac{\sin a\pi}{a\pi} = \prod_{n=1}^{\infty} \left(1 - \frac{a^2}{n^2}\right), \quad 0 < a < 1,$$

Please, turn over!

and, by means thereof, Wallis' product formula

$$\frac{\pi}{2} = \frac{2}{1}\frac{2}{3}\frac{4}{3}\frac{4}{5}\frac{6}{5}\frac{6}{7}\frac{8}{7}\frac{8}{9}\cdots .$$