Lund University, Numerical Analysis **Spring Term 2, 2019**
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Finite Volume Methods

Assignment 2

Problem 1

Definition 1 (Weak solution) *A solution of a scalar conservation law* $u_t + f(u)_x = 0$ *is called a weak solution if*

$$\int_0^\infty \int_{-\infty}^\infty (\phi_t u + \phi_x f(u)) dxdt = -\int_{-\infty}^\infty \phi(x,0)u(x,0)dx$$

for all functions $\phi \in \mathcal{C}_0^1(\mathbb{R} \times \mathbb{R})$.

Now consider the following Burger's equation problem:

$$u_t + uu_x = 0,$$

$$u(0,x) = \begin{cases} u_L & x \le 0, \\ u_R & x > 0. \end{cases}$$

Note: A problem with piecewise constant initial data with one jump is called a Riemann problem.

Prove: The following solutions are weak solutions of the above problem in case $u_L < u_R$:

$$u_1(x,t) = \begin{cases} u_L & x \le st, \\ u_R & x > st. \end{cases}$$

with $s = (u_L + u_R)/2$.

$$u_2(x,t) = \begin{cases} u_L & x < u_L t, \\ x/t & u_l t \le x \le u_r t, \\ u_R & x > u_R t. \end{cases}$$

The second one is called a rarefaction wave.

Problem 2

Perform a von Neumann stability analysis of the upwind scheme $u_x(x_i) \approx \frac{u_i - u_{i-1}}{\Delta x}$ combined with the explicit Euler method with fixed step sizes for the linear advection equation with $a > 0$.

Problem 3
Consider the Burger's equation

$$u_t + uu_x = 0, \quad x \in [-1,1], t \in [0,\infty],$$

$$u(0,x) = \begin{cases} 1 & x \leq 0, \\ 0 & x > 0, \end{cases}$$

$$u(t,-1) = 1.$$

a) Discretize this using the explicit Euler method with constant mesh width Δt in time and an upwind finite difference method with constant mesh width Δx in space. Specifically, use the approximation

$$(uu')(x_i) \approx u_i(u_i - u_{i-1})/\Delta x.$$

- Program this in a language of your choice and visualize the numerical solutions for $t = 0$, $t = 1$ and $t = 2$ for $\Delta x = 1/100$ and a stable Δt. What do you see?
- Refine the mesh and the time step. What happens?

b) Change the space discretization by instead of approximating $(uu')(x_i)$, approximate

$$(\frac{1}{2}u^2)'(x_i)$$

appropriately. What do you observe?

Return: Tuesday, April 9th, in class