LUND UNIVERSITY

Written Examination
Distribution Theory
Monday June 2, 2014
Time: 08.00–13.00

Centre for Mathematical Sciences
Mathematics, Faculty of Science

No aids allowed. Use the distributed paper sheets and write only on one side. Fill in the cover sheet completely and initialize every sheet. Write legibly (in Swedish or English). Motivate your conclusions clearly and concisely; draw a picture if that helps to clarify your argument.

Test results: *Posted Tuesday June 3, before 16.00.*

1. Determine all $u \in \mathcal{D}'(\mathbf{R})$ that satisfy $x(u' + u) = \delta_0 - \delta_1$.

2. Compute the Fourier transform of

a) the function $x_1 \sin(x_2)/(1 + x_1^2)$ on $\mathbf{R}^2$

b) the distribution $u \in \mathcal{S}'(\mathbf{R}^2)$ defined by $u(\phi) = \iint_{|x_1| \le 1} e^{-|x_2|}\phi(x_1, x_2)\, dx_1 dx_2$.

3. Define

$$F = \{\, f * f : f \in \mathcal{E}'(\mathbf{R}^n) \,\} \subseteq \mathcal{E}'(\mathbf{R}^n)$$

Show that $F \neq \mathcal{E}'(\mathbf{R}^n)$.

4. Let $u_n(x) = \operatorname{sgn}(x) n^2 e^{-n|x|}$ for $n = 1, 2, \ldots$, where $\operatorname{sgn}(x)$ is the sign of $x \in \mathbf{R}$. Show that u_n has a limit in $\mathcal{D}'(\mathbf{R})$ as $n \to \infty$ and compute this limit.

5. Define the distribution $u = \sum_{k=1}^{\infty} k\delta_{1/k^2} \in \mathcal{D}'(\mathbf{R}_+)$, where $\mathbf{R}_+ = \{\, x \in \mathbf{R} : x > 0 \,\}$ and δ_a is the Dirac measure at $x = a$. Show that u can be extended to a distribution $u \in \mathcal{D}'(\mathbf{R})$, so that $u = 0$ when $x < 0$, and determine such an extension explicitly.

6. Assume that $u(x)$ is a continuous and integrable function on $\mathbf{R}^n$ such that $\hat{u}$ has compact support. Show that $\sum_{a \in \mathbf{Z}^n} u(a)$ is absolutely convergent. (Hint: use that $\hat{u} = \widehat{\varphi}\hat{u}$ for some $\varphi \in \mathcal{S}$.)