Review of Distribution Theory

Chapter 2: Show how one can construct non-trivial test functions with support in an arbitrary subset.

Show how one can construct a partition of unity subordinate to a given open cover.

Chapter 3: Show that the mapping *test:* $C(X) \mapsto \mathcal{D}'(X)$ is injective.

Show that a distribution can be estimated by some C^k norm on any given compact set.

Show that a distribution of order k has a unique extension to a continuous linear form om C_0^k.

Show that every positive distribution is a positive measure.

Chapter 4: Show that the equation $u' = f \in \mathcal{D}'(X)$ has a unique solution $u \in \mathcal{D}'(X)$ modulo constants.

Chapter 5: Show that if $u_k \to u$ in $\mathcal{D}'(X)$, then $\partial^\alpha u_k \to \partial^\alpha u$ in $\mathcal{D}'(X)$ for any α.

Chapter 7: Show that if $u \in \mathcal{D}'(X)$ and $u = 0$ near any given point, then $u \equiv 0$.

Show that $\mathcal{D}'(X)$ forms a sheaf over X.

Chapter 8: Show that the restriction map $\mathcal{E}'(X) \mapsto \mathcal{D}'(X)$ is injective.

Show that $u \in \mathcal{E}'(X)$ if and only if the support of u is compact.

Show that if $\operatorname{supp} u = \{ a \}$ then u is a sum of derivatives of the Dirac measure at a.

Chapter 9: Show that $\mathcal{E}'(X)$ is dense in $\mathcal{D}'(X)$.

Show that an ODE with smooth coefficients can be solved in $\mathcal{D}'(X)$, modulo smooth solutions to the homogeneous equation.

Show that if $u \in \mathcal{D}'(X)$ and $\psi u = 0$ for some $\psi \in C^\infty$, then $\operatorname{supp} u \subseteq \psi^{-1}(0)$.

Chapter 10: Show how to define the push-forward $\Phi_* u$ for a smooth and proper Φ.

Show how to define the pull-back $\Phi^* u$ when Φ is a diffeomorphism.

Show that $u \in \mathcal{D}'(X)$ is homogeneous of degree a if and only if $\langle x, \partial u \rangle = au$.

Chapter 11: Show that the convolution of a distribution and a smooth function is well defined and a smooth function, when at least one of the supports is compact.

Show that test functions are dense in $\mathcal{D}'(X)$.

Show that tensor products of distributions are well defined.

When is a convolution $u * v$ of distributions u and v well defined?

Show that the suppport of a convolution is contained in the sum of the supports of the factors.

Show that the singular suppport of a convolution is contained in the sum of the singular supports of the factors.

Chapter 12: What is a fundamental solution and a parametrix to a constant coefficient PDE?

Show that a constant coefficient PDE is hypoelliptic if it has a parametrix E such that $\operatorname{sing\,supp} E = \{ 0 \}$.

Chapter 14: Show that test functions are dense among temperate test functions.

Show that the Fourier transform is a bijection.

Show that the Fourier transform of a distribution with compact support is an analytic function.

Show that the convolution $u * v$ is well defined when $u \in \mathcal{S}'(X)$ and $v \in \mathcal{E}'(X)$.

Chapter 15: Show that a mapping with a right semiregular distribution kernel extends to a continuous linear mapping $\mathcal{E}' \mapsto \mathcal{D}'$.

What does it means that a mapping $C^\infty \mapsto \mathcal{D}'$ is pseudo-local?

Chapter 17: Show that if $P(D)$ is elliptic then it is hypoelliptic.

Show that if $P(D)$ is elliptic then it has a fundamental solution.