Lund University | **Distribution Theory**
Centre for Mathematical Sciences | **Exercise sheet**

1. Let
$$h(x) = \int_{|x|}^{1} e^{-1/t(1-t)}\,dt \qquad \text{when } |x| < 1,\ h(x) = 0 \text{ då } |x| \geq 1.$$
Show that $h(x) + h(x-1) = h(0)$ when $0 \leq x \leq 1$. Let
$$\varphi(x) = \sum_{k=-n}^{n} h(x-k)/h(0).$$
Show that $\varphi \in C_0^\infty(\mathbf{R})$ with support where $|x| \leq n+1$ such that $\varphi(x) = 1$ when $|x| \leq n$.

2. Let $H(t)$ be the Heaviside function (or step function), which is equal to 1 when $t > 0$ and equal to 0 when $t \leq 0$. Let $\chi_h(t) = (H(t+h) - H(t-h))/2h$. Show that if $f \in C_0$ then
$$f * \chi_h(t) = \int f(t-s)\chi_h(s)\,ds$$
is a C_0^1 regularization of f, i.e., $f * \chi_h \in C_0^1$, $f * \chi_h \to f$ uniformly and $\operatorname{supp} f * \chi_h \to \operatorname{supp} f$ when $h \to 0$.

3. a) Show that the following expressions with $\varphi \in C_0^\infty(\mathbf{R})$ are equal (and finite)
$$\mathrm{Pf} \int_{-\infty}^{\infty} \frac{\varphi(x)}{x^2}\,dx \overset{\text{def}}{=} \lim_{\varepsilon \to 0} \left(\int_{|x|>\varepsilon} \frac{\varphi(x)}{x^2}\,dx - 2\frac{\varphi(0)}{\varepsilon} \right) = -\int_{-\infty}^{\infty} \log|x|\varphi''(x)\,dx$$
and
$$\mathrm{Pf} \int_{0}^{\infty} \frac{\varphi(x)}{x^2}\,dx \overset{\text{def}}{=} \lim_{\varepsilon \to 0} \left(\int_{\varepsilon}^{\infty} \frac{\varphi(x)}{x^2}\,dx - \frac{\varphi(0)}{\varepsilon} + \varphi'(0)\log\varepsilon \right) = \varphi'(0) - \int_{0}^{\infty} \log|x|\varphi''(x)\,dx.$$

b) Show that
$$\mathrm{Pf}\ \frac{1}{x^2} : \varphi \to \mathrm{Pf} \int_{-\infty}^{\infty} \frac{\varphi(x)}{x^2}\,dx$$
and
$$\mathrm{Pf}\ \frac{H(x)}{x^2} : \varphi \to \mathrm{Pf} \int_{0}^{\infty} \frac{\varphi(x)}{x^2}\,dx$$
are distributions on $\mathbf{R}$ and determine their orders.
Here Pf stands for ”partie finie” – finite part according to Hadamard.

c) Define $\mathrm{Pf}\ \frac{H(x)}{x^{5/2}}$.

4. Compute the following limits in $\mathcal{D}'(\mathbf{R})$:

a) $\lim_{t\to\infty} \frac{t}{1+t^2x^2}$,

b) $\lim_{t\to 0} t^{-1/2}e^{-x^2/4t}$,

c) $\lim_{t\to\infty} t^2 x\cos(tx)$,

d) $\lim_{t\to\infty} t^2|x|\cos(tx)$.

5. Assume that the function $f(x)$ is continuous outside the origin and that $|x|^m f(x)$ is locally integrable. Show that f has an extension to $\mathcal{D}'(\mathbf{R}^n)$ which has order m. What is the difference between two extensions?

6. Determine all solutions $u \in \mathcal{D}'(\mathbf{R})$ to the differential equations

$$xu' + u = 0$$
$$x^2u' + u = 0.$$

Hint: use an integrating factor when $x \neq 0$.

7. Show that $\delta_0^{(\alpha)} \in \mathcal{D}'(\mathbf{R}^n)$ is homogeneous of degree $-|\alpha| - n$.

8. a) Show that

$$\varphi \mapsto \int_0^\infty x^{k-b}\varphi^{(k)}(x)\,dx \qquad b < 1+k$$

is a distribution which is homogeneous of degree $-b$.

b) Show that

$$\varphi \mapsto (-1)^k \int_0^\infty \varphi^{(k)}(x)\log x\,dx/(k-1)! \qquad k \in \mathbf{N}$$

is not a homogeneous distribution.

c) Show that

$$\varphi \mapsto (-1)^k \int_{-\infty}^\infty \varphi^{(k)}(x)\log|x|\,dx/(k-1)! \qquad k \in \mathbf{N}$$

is a distribution which is homogeneous of degree $-k$.

9. Show that if $u \in \mathcal{D}'(\mathbf{R}^n)$ is homogeneous of degree μ, then $\partial^\alpha u$ is homogeneous of degree $\mu - |\alpha|$.

10. Compute $f * f * \cdots * f$ (with n factors) when

a) $f(t) = H(t)$,

b) $f(t) = e^{-t}H(t)$.

11. Let f_a be the characteristic function of the interval $(0, a) \subset \mathbf{R}$, where $a > 0$. Find a distribution u_a with support in $\overline{\mathbf{R}}_+$ so that $u_a * f_a = \delta_0$.

12. Determine the constant A so that

$$E(x,t) = \begin{cases} A & \text{when } \nu^2 t^2 - x^2 \geq 0 \text{ and } t \geq 0 \\ 0 & \text{elsewhere} \end{cases}$$

becomes a fundamental solution to the differential operator $\nu^{-2}\partial_t^2 - \partial_x^2$ on $\mathbf{R}^2$. (Hint: choose $\nu t \pm x$ as new variables.)

13. Determine an entire analytic function $F(z)$ so that

$$E(x,y) = \begin{cases} F(cxy) & \text{when } x \geq 0 \text{ and } y \geq 0 \\ 0 & \text{elsewhere} \end{cases}$$

becomes a fundamental solution to the differential operator $\partial_x \partial_y - c$ on $\mathbf{R}^2$ for $c \in \mathbf{C}$. (It suffices to determine the power series expansion for F.)

14. Compute the Fourier transforms of the following temperate distributions on $\mathbf{R}$:

a) $e^{-\eta t} H(t)$, $\eta > 0$ b) $H(t)$ c) $\operatorname{sgn} t$ d) 1 e) $e^{i\alpha t}$ f) $\sin \alpha t$

g) $e^{i\alpha|t|}$ h) $\sin \alpha|t|$ i) $|t|$ j) $tH(t)$ k) $\frac{1}{1+t^2}$ l) $\arctan t$

15. Show that if u and $v \in \mathcal{E}'(\mathbf{R}^n)$ such that $u * v \equiv 0$, then u or $v \equiv 0$. Does this hold if $u \in \mathcal{E}'(\mathbf{R}^n)$ and $v \in \mathcal{S}'(\mathbf{R}^n)$?

16. For which $u \in \mathcal{S}'(\mathbf{R}^n)$ do there exist $f \in \mathcal{S}(\mathbf{R}^n)$ so that $u = u * f$?

17. Assume that f is a continuous function from $\mathbf{R}$ to $\mathbf{R}$. Which operator has the distribution kernel $\partial H(y - f(x))/\partial y$?

18. What is the kernel to the operator

$$\mathcal{K}\varphi(x) = \varphi(x) + \int_{\mathbf{R}} a(x,y)\varphi'(y)\,dy, \qquad \varphi \in C_0^\infty(\mathbf{R}),$$

where $a \in C(\mathbf{R}^2)$?

19. The function f is integrable on $\mathbf{R}^n$ such that $f * f = f$. Determine f.

20. Let u be a temperate distribution whose Fourier transform has compact support. Show that $u \in C^\infty$ and that for any j there exists a function $f_j \in \mathcal{S}$ so that $\partial_{x_j} u = u * f_j$.