Reconstruction and Global Optimization

- Framework
- Convex Optimization
- Triangulation
- The Bisection Algorithm
Under the assumption that image points are corrupted by Gaussian noise, minimize the reprojection error.

The reprojection error

In regular coordinates \((x = (x, y))\) the projection is

\[
\begin{pmatrix}
    p^1_x & p^2_x \\
    \frac{p^2_x}{p^3_x} & \frac{p^2_x}{p^3_x}
\end{pmatrix}.
\]

\(p^1, p^2, p^3\) are the rows of \(P\).

The reprojection error is

\[
\left\| \left( x - \frac{p^1_x}{p^3_x}, y - \frac{p^2_x}{p^3_x} \right) \right\|^2.
\]
Framework: Affine Projective Estimation

\[ r_i(x) = \frac{(a_i^T x + \tilde{a}_i)^2 + (b_i^T x + \tilde{b}_i)^2}{(c_i^T x + \tilde{c}_i)^2}, \quad c_i^T x + \tilde{c}_i > 0. \]

Solve either the projective least-squares problem

\[
\min_{\{x; c_i^T x + \tilde{c}_i > 0, \ \forall i\}} \sum_i r_i(x),
\]

or the min-max problem (easier)

\[
\min_{\{x; c_i^T x + \tilde{c}_i > 0, \ \forall i\}} \max_i r_i(x).
\]
Quotients of Affine Functions

If

\[ P = \begin{bmatrix} -A^1 & t_1 \\ -A^2 & t_2 \\ -A^3 & t_3 \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} X \\ 1 \end{bmatrix}. \]

Then

\[ \left\| \left( x - \frac{P^1X}{P^3X}, y - \frac{P^2X}{P^3X} \right) \right\|^2 = \right. \]

\[ \left. \left\| \left( \frac{x_1(A^3X + t_3) - (A^1X + t_1)}{A^3X + t_3}, \frac{x_2(A^3X + t_3) - (A^2X + t_2)}{A^3X + t_3} \right) \right\|^2 \right. \]

Least squares problem with quotients of affine functions. If either \( A \) or \( X \) is known!
Framework Examples: Triangulation

**Given**
- Image data (2D points - $x$)
- Cameras ($P$)

**Estimate**
- Structure (3D points - $X$)

Quotients of affine functions in $X$!
Framework Examples: Resection (uncalibrated)

Given
- Image data (2D points - $x$)
- Structure (3D points - $X$)

Estimate
- Cameras ($P$)

Quotients of affine functions in $A$, $t$!

$$
\begin{bmatrix}
\frac{x_1(A^3X+t_3)-(A^1X+t_1)}{A^3X+t_3} \\
\frac{x_2(A^3X+t_3)-(A^2X+t_2)}{A^3X+t_3}
\end{bmatrix}^2
$$
Framework Examples: SfM with Known Orientations

Given
- Image data (2D points - $u$)
- Camera orientations ($A$)

Estimate
- Structure ($U$), Positions ($t$).

\[ \left\| \frac{x_1(A^3X + t_3) - (A^1X + t_1)}{A^3X + t_3} - \frac{x_2(A^3X + t_3) - (A^2X + t_2)}{A^3X + t_3} \right\|^2 \]

Quotients of affine functions in $X$ and $t$!
Why use the projective least-squares formulation?

\[
\min_{\{x; c_i^T x + \bar{c}_i > 0, \ \forall i\}} \sum_i r_i(x),
\]

- Geometrically meaningful goal function (minimize reprojection error).
- Statistically optimal (under the assumption of Gaussian noise).

Why the min-max problem?

\[
\min_{\{x; c_i^T x + \bar{c}_i > 0, \ \forall i\}} \max_i r_i(x).
\]

- Geometrically meaningful goal function (minimize reprojection error).
- Easier to minimize due to convexity properties.
Global Optimization

See lecture notes.
Global Optimization

Checking if there is

\[ \mathbf{X} \in \bigcap_{i \in I} \{ \mathbf{X}; \quad r_i(\mathbf{X}) \leq \epsilon^2, \quad P^3_i \geq \delta \} \]

is a convex problem:

\[
\min_{s, \mathbf{X}} \quad s \\
\text{such that} \quad \| (x_i P^3_i - P^1_i) \mathbf{X}, (y_i P^3_i - P^2_i) \mathbf{X} \| \leq \epsilon P^3_i \mathbf{X} + s, \quad \forall i \in I \\
\mathbf{P}_i \mathbf{X} \geq \delta, \quad \forall i \in I.
\]

If \( s > 0 \) then the set is empty!
Global Optimization: Triangulation
The 3D point must lie in the intersection of the cones.
Reduce the size of the cones $\iff$ lower the permitted error. As long as there is a point in the intersection.
No point in the intersection.
Global Optimization: Triangulation

Algorithm

Minimizes the maximal reprojection error. Finds the smallest possible $\epsilon$ for which there is a solution $X$ with all reprojection errors is less than $\epsilon$. That is, solves

$$\min_X \max_i r_i(X).$$

1. Let $\epsilon_l$ and $\epsilon_u$ be lower and upper bound on the optimal error.
2. Check if there is a solution such that

   $$r_i(X) \leq \frac{\epsilon_u + \epsilon_l}{2}, \quad \forall i$$

   (convex optimization problem).
3. If there is set $\epsilon_u = \frac{\epsilon_u + \epsilon_l}{2}$, otherwise set $\epsilon_l = \frac{\epsilon_u + \epsilon_l}{2}$.
4. If $\epsilon_u - \epsilon_l > tol$ (some predefined tolerance) goto 2.
Global Optimization

Generalizations

Works for other problems as well:

- Computing camera matrix given 3D-points and projections.

\[ x_i \sim P \begin{bmatrix} X_i \end{bmatrix} \]

- Homography estimation.

\[ y_i \sim H \begin{bmatrix} x_i \end{bmatrix} \]

- Structure and motion if camera orientations are known.

\[ x_{ij} \sim \begin{bmatrix} R_i & t_i \end{bmatrix} \begin{bmatrix} X_j \end{bmatrix} \]