Calculus of Variations — Lecture 6

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Exercises

1. For vector valued problems, show that if $\boldsymbol{y} = \boldsymbol{y}(x) = (y_1(x), \dots, y_n(x))$ is *an extremal* of a functional with Lagrangian $F = F(x, \boldsymbol{y}, \boldsymbol{y}')$, then

$$F - \sum_{i=1}^{n} y_i' \frac{\partial F}{\partial y_i'} = F_x \, .$$

In particular, if F is independent of x then we have "conservation of energy", $F - \boldsymbol{y}' \cdot F_{\boldsymbol{y}'} = -E$ for some real constant E. (Here a natural short hand for the velocity and the partial gradient of F was used. We write "$-E$" because the conserved expression is actually minus the energy if a mechanical system is considered.) In the computation you mmay assume that F and y are as smooth as you please.

2. Suppose that we are to minimize a standard functional,

$$J[y] = \int_a^b F(x, y(x), y'(x))\, dx$$

over all functions in C^1 which satisfies the boundary conditions $y(b) = y(a)$. Derive the corresponding natural boundary condition. Show that if $y' \mapsto F_{y'}(x, y, y')$ is a strictly monotone function, then any extremal of this problem may be thought of as periodic C^1-function.

3. Derive the natural boundary conditions for a solution $\phi = \phi(x)$ to the problem

$$\min_{y \in C^1} J[y] := \int_a^b F(x, y(x), y'(x))\, dx$$

using the C_1-theory. (In the lectures we assumed that $\phi \in C^2$.)