Calculus of Variations — Lecture 2

28 January 2019, Niels Chr Overgaard

Exercises

1. Use Euler's equation to solve the variational problem: Minimize

$$I[y] = \int_0^2 y'(x)^2 + 2y(x)\,dx\,,$$

where the minimum is taken over all functions y which are (twice) continuously differentiable and satisfies the end point-conditions $y(0) = 0$ and $y(2) = 4$.

2. Consider the problem of minimizing the functional

$$J[y] = \int_0^1 y(x)^2 y'(x)^2\,dx$$

subject to the end point conditions $y(0) = 0, y(1) = 1$. Determine its admissible extremals. (Cf. Problem 3 from the exercise set to the first lecture.)

3. Consider the following three variational problems. In each case, write down the corresponding Lagrange function $F = F(x, y, z)$ (and specify its natural domain of definition), derive the Euler equation, find the set of extremals and, finally, determine which extremals are admissible. (Observe that different formulations are used for each of these problems.)

a) Minimize the functional

$$J[y] = \frac{1}{2}\int_0^{\ln(1+\sqrt{2})} y'(x)^2 + y(x)^2\,dx$$

subject to $y \in C_1$ and the end point-conditions $y(0) = 0$, $y(\ln(1+\sqrt{2})) = 1$.

b) Minimize the integral

$$I[B] = \int_0^1 \frac{B(x)}{B'(x)}\,dx$$

over functions $B \in \mathscr{A}$. Here

$$\mathscr{A} := \left\{ B \in C_1 : B(x) > 0, B'(x) > 0 \text{ for all } 0 < x < 1, \text{ and } B(0) = 1, B(1) = 3 \right\}$$

is the set of *admissible functions* for the problem.

c) Find the minimum for the "action" of the harmonic oscillator:

$$A[u] = \frac{1}{2}\int_0^{\pi/2} \dot{u}(t)^2 - u(t)^2\,dt$$

subject to $u \in C_1$ and the end point-conditions $u(0) = 0$, $u(\pi/2) = 1$. What happens if we change the right interval end point to $t = \pi$?

4. Determine the admissible extremals of the functionals below subject to the end point conditions $y(0) = 0$ and $y(1) = 1$.

a) $J[y] = \int_0^1 yy'\,dx$.

b) $J[y] = \int_0^1 y^2 + 2xyy'\,dx$.

c) $J[y] = \int_0^1 \frac{x+yy'}{\sqrt{1+x^2+y^2}}\,dx$.

What are the extremals (i.e. solutions of Euler's equation)? Can the above examples be generalized?

5. Determine the minimum value of the functional

$$J[y] = \int_0^1 -xyy'\,dx$$

among the functions in C^1 which satisfies the end point conditions $y(0) = 0$ and $y(1) = 1$. Does the problem have admissible extremals?

6. Find the admissible extremals for the problem of minimizing the functional

$$J[y] = \int_0^b \frac{1}{2}(y'(x) - x)^2\,dx$$

subject to the end point conditions $y(0) = 0$ and $y(b) = \beta$, where $b > 0$ och $\beta \in \mathbf{R}$ are given numbers.

Guess the minimum of the above problem. Write the answer as a function $S = S(b, \beta)$ of the coordinates of the right end point (b, β). Can the guess be verified?